Tohoku Math. J.
51 (1999), 193–203

# $L_p$ AND BESOV MAXIMAL ESTIMATES FOR SOLUTIONS TO THE SCHRÖDINGER EQUATION

SEIJI FUKUMA AND TOSINOBU MURAMATU

(Received October 31, 1997)

**Abstract.** Precise results on $L_p$ and Besov estimates of the maximal function of the solutions to the Schrödinger equation are given. These results contain an improvement of the theorem in Sjölin [10].

**1. Introduction.** It is well-known that the solution to the Schrödinger equation

$$\frac{\partial u}{\partial t} = -i\Delta u, \quad u(0,x) = f(x), \qquad (x \in \boldsymbol{R}^n,\ t \in \boldsymbol{R}) \tag{1.1}$$

is given by

$$u(t,x) = c_n \iint e^{i(x-y)\xi + it|\xi|^2} f(y) d\xi dy .$$

In this note we shall consider estimates of $L_2$-norm and the Besov type norm of integrals of this kind by means of the Besov norm of $f$, and give $L_p$-estimates of their maximal functions.

Our first results are the following two theorems:

THEOREM 1. *Let $\sigma$ be a positive number, $I = (0,1)$, $\gamma > 1$ and let $1 \le q \le \infty$. Assume that $h(t,\xi)$ is real-valued, measurable, and $C^\infty$ in $t$ and the inequality*

$$\left|\frac{\partial^k h(t,\xi)}{\partial t^k}\right| \le C_k(1 + |\xi|^{k\gamma}) \tag{1.2}$$

*holds for any positive integer $k$, where $C_k$ is a constant independent of $t$ and $\xi$. Then, the operator $T_1$ defined by*

$$T_1 f(t,x) = c_n \iint_{\boldsymbol{R}^n} e^{i(x-y)\xi + ih(t,\xi)} f(y) d\xi dy , \tag{1.3}$$

*where $c_n = (2\pi)^{-n}$, is bounded from $B^{\gamma\sigma}_{2,q}(\boldsymbol{R}^n)$ to $B^{\sigma}_{2,q}(I; L_2(\boldsymbol{R}^2))$.*

THEOREM 2. *Let $h$ be a real-valued function satisfying the condition* (1.2). *Then, the operator $T_1$ defined by* (1.3) *is bounded from $B^{\gamma/2}_{2,1}(\boldsymbol{R}^n)$ to $L_2(\boldsymbol{R}^n; L_\infty(I))$, i.e.,*

$$\left(\int_{\boldsymbol{R}^n} \|T_1 f(x,\cdot)\|^2_{L_\infty(I)} dx\right)^{1/2} \le C\|f\|_{B^{\gamma/2}_{2,1}} . \tag{1.4}$$

For the operator of the type (1.5) below acting on Sobolev spaces $H^s$, there are several papers. Carbery [1] and Cowling [2] have prove that $T_2$ is bounded from $H^s(\boldsymbol{R}^n)$ to $L_2(I; L_2(\boldsymbol{R}^n))$ for $s > a/2$, and Theorem 2 is an improvement of their results. P. Sjölin [10]

1991 *Mathematics Subject Classification.* Primary 42B25; Secondary 35Q55.
*Key words and phrases.* Maximal function, Besov norm, Schrödinger equation.