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ELASTICAE WITH CONSTANT SLANT IN THE COMPLEX PROJECTIVE PLANE AND NEW EXAMPLES OF WILLMORE TORI IN FIVE SPHERES

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Abstract. We exhibit a reduction of variables criterion for the Willmore variational problem. It can be considered as an application of the Palais principle of symmetric criticality. Thus, via the Hopf map, we reduce the problem of finding Willmore tori (with a certain degree of symmetry) in the five sphere equipped with its standard conformal structure, to that for closed elasticae in the complex projective plane. Then, we succeed in obtaining the complete classification of elasticae with constant slant in this space. It essentially consists in three kinds of elasticae. Two of them correspond with torsion free elasticae. They lie into certain totally geodesic surfaces of the complex projective plane and their slants reach the extremal values. The third type gives a two-parameter family of helices, lying fully in this space. A nice closure condition, involving the rationality of one parameter, is obtained for these helices. Hence, we get three associated families of Willmore tori in the standard five sphere. They are Hopf map liftings of the above mentioned families of elasticae. The method also works for a one-parameter family of conformal structures on the five sphere, which defines a canonical deformation of the standard one.

1. Introduction. Let M be an immersed compact surface (throughout this paper surfaces are assumed to be compact) into a Riemannian manifold $\tilde{M}$. We denote by α and S the mean curvature function of M and the sectional curvature function of $\tilde{M}$ with respect to the tangent space of M, and define

$$\mathcal{W}(M)=\int_M(\alpha^2+S)dv\,.$$

This functional is an invariant under conformal changes of the metric of $\tilde{M}$ and the critical points of $\mathcal{W}$ are called *Willmore surfaces* ([6]).

Minimal surfaces of a sphere are obvious examples of Willmore surfaces in such a sphere. However, N. Ejiri [8], answering to a problem of J. L. Weiner [16], gave an example of a non-minimal Willmore flat torus in S^5. Later, U. Pinkall [15], using a nice description for the Hopf fibration of S^3 onto S^2 (both unit spheres), gave an infinite family of unstable non-minimal Willmore surfaces in S^3 which can be obtained

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