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ON MUMFORD'S CONSTRUCTION OF DEGENERATING ABELIAN VARIETIES

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Abstract. For a one-dimensional family of abelian varieties equipped with principal theta divisors a canonical limit is constructed as a pair consisting of a reduced projective variety and a Cartier divisor on it. Properties of such pairs are established.

CONTENTS

Introduction. Assume that we are given a 1-parameter family of principally polarized abelian varieties with theta divisors. By this we will mean that we are in one of the following situations:

1. $\mathcal{R}$ is a complete discrete valuation ring (DVR, for short) with the fraction field K, $S = \operatorname{Spec}\mathcal{R}$, $\eta = \operatorname{Spec}K$ is the generic point, and we have an abelian variety G_η over K together with an effective ample divisor Θ_η defining a principal polarization; or

2. we have a projective family (G, Θ) over a small punctured disk D^0_ε.

In this paper we show that, possibly after a finite ramified base change, the family can be completed in a simple and absolutely canonical manner to a projective family (P, Θ) with a relatively ample Cartier divisor Θ over S, resp. D_ε. Moreover, this construction is stable under further finite base changes. We give a combinatorial description of this family and its central fiber (P_0, Θ_0) and study their basic properties. In particular, we prove that P_0 is reduced and Cohen-Macaulay and that $H^i(P_0, \mathcal{O}(d\Theta_0))$, $d \geq 0$ are the same as for an ordinary PPAV (principally polarized abelian variety).

Existence of such construction has profound consequences for the moduli theory. Indeed, with it one must expect that there exists a canonical compactification $\bar{A}_g$ of the moduli space A_g of PPAVs, similar to the Mumford-Deligne compactification of the moduli space of curves. Without it, one has to believe that there is no single "best" geometrically meaningful

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