Real Analysis Exchange
Vol. 19(1), 1993/94, pp. 50–51

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EXPANSION VIA LEGENDRE FUNCTIONS

An analogue of Fourier series, well known in potential theory, is Laplace's expansion of a function f in a series of Legendre polynomials P_n:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \text{ for } -1 < x < 1,$$

where

$$c_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(t) P_n(t) dt.$$

Szegö's "Orthogonal Polynomials" has generalizations of this to Jacobi and other polynomials. At Baltimore I described a different generalization:

$$\left.\begin{aligned} f(x) &= \textstyle\sum_{n=-\infty}^{\infty} a_n P_{\nu+n}^{\mu}(x) \text{ for } -1 < x < 1, \\ \text{where} \qquad a_n &= (-1)^n \tfrac{\nu+n+\frac{1}{2}}{2\cos\nu\pi} \textstyle\int_{-1}^{1} f(t) P_{\nu+n}^{-\mu}(-t) dt. \end{aligned}\right\} \tag{1}$$

Here $P_\nu^\mu(t)$ are not polynomials, but solutions of the "associated" Legendre's equation with parameters μ and ν not necesarily integers; for $\mu = 0$ and $\nu = n$, an integer, they reduce to Legendre polynomials.

The requirements for (1) are, roughly, that f be slightly more than integrable on $[-1, 1]$ and slightly more than continuous at x. However, familiar Fourier series theory allows f to have a simple discontinuity at x such that $\{f(x+0) + f(x-0)\}/2 = f(x)$, provided that it also has bounded variation on a neighborhood of x.

To see whether an extension of (1) allowing discontinuity is possible, I studied the Abel summability of the series, in the (slightly extended) sense:

$$\text{if } A(r) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n P_{\nu+n}^{\mu}(x), \text{ does } \lim_{r \to 1^-} A(r) \text{ exist?}$$

Putting $x = \cos\omega$, $g(\theta) = f(\cos\theta)\sin^{-\mu}\theta$, and using Mehler's Integral

$$P_\nu^\mu(\cos\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin^\mu \omega}{\Gamma(\frac{1}{2} - \mu)} \int_0^\omega \frac{\cos\left(\nu + \frac{1}{2}\right)\phi}{(\cos\phi - \cos\omega)^{\frac{1}{2}+\mu}} d\phi,$$