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UNIONS OF PRODUCTS OF INDEPENDENT SETS

M. Laczkovich asked whether it is possible to find a Lebesgue measurable set $H \subset [0,1] \times [0,1]$ which is of full two dimensional measure, that is $\lambda_2(H) = 1$, and which for any $\epsilon > 0$ contains product sets of the form $E \times E$ with $\lambda_1(E) > \frac{1}{2} - \epsilon$, but there is no product set $S \times S \subset H$ with $\lambda_1(S) = \frac{1}{2}$. M. Laczkovich also suggested a construction for such a set H. Take stochastically independent sets $E_i \subset [0,1]$, $i = 1, 2, \ldots$ with $\lambda_1(E_i) < 1/2$, and $\lambda_1(E_i) \to 1/2$ and set $H = \cup_{i=1}^{\infty} E_i \times E_i$. In [1] we show that this construction works.

It follows from §4 of [2] that if (E_i) is a stochastically independent sequence of Lebesgue measurable sets with $\sup_{i \in N} \lambda_1(E_i) \leq 1/2$, and S is a Lebesgue measurable set such that $S \times S \subset \cup_{i=1}^{\infty} E_i \times E_i$ then $\lambda_1(S) \leq 1/2$. Our result shows that from $\lambda_1(E_i) < 1/2$ it follows that $\lambda_1(S) < 1/2$.

As it is shown in §3 of [2] the number 1/2 has a special role. For any $\epsilon \in (\frac{1}{2}, 1)$ and $\eta \in (0,1)$ one can choose a sufficiently large n, such that for any finite sequence of independent sets $E_1, \ldots, E_n$ with $\lambda_1(E_i) = \epsilon$ $(i = 1, \ldots, n)$ there exists a measurable set S with $S \times S \subset \cup_{i=1}^{n} E_i \times E_i$ and $\lambda_1(S) = \eta$.

The main results of [1] are

Theorem 1 *There exists an F_σ set $H \subset [0,1] \times [0,1]$ of full λ_2-measure in $[0,1] \times [0,1]$ such that for any $\epsilon > 0$ there exists a closed set $E \subset [0,1]$ satisfying $\lambda_1(E) > \frac{1}{2} - \epsilon$, and $E \times E \subset H$ but there is no Lebesgue measurable $S \subset [0,1]$ with $\lambda_1(S) = 1/2$ and $S \times S \subset H$.*

Theorem 2 *If (E_i) is a stochastically independent sequence of Lebesgue measurable subsets of $[0,1]$ with $\lambda_1(E_i) < 1/2$ then there is no Lebesgue measurable set $S \subset [0,1]$ such that $S \times S \subset \cup_{i=1}^{\infty} E_i \times E_i$ and $\lambda_1(S) = 1/2$.*

It is clear that Theorem 1 is an easy Corollary of Theorem 2. However to answer the original question of M. Laczkovich a very special sequence of independent sets provides an easier proof (Lemma 2 and Theorem 1 of [1]). In

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