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A Restricted Symmetric Derivative for Continuous Functions of Two Variables

In this paper we are concerned with symmetric differences for real valued functions defined on the plane R^2. If $f(x,y)$ is defined on R^2, the symmetric difference at (x,y) is

$$\Delta f(x,y;h,k) = f(x+h,y+k)+f(x-h,y-k)-f(x+h,y-k)-f(x-h,y+k).$$

This difference is used to generalize second order partial derivatives as $\frac{\partial^2 f}{\partial x \partial y} = \lim_{h,k\to 0} \frac{\Delta f(x,y;h,k)}{4hk}$ if f is C^2, where $\lim_{h,k\to 0} \frac{\Delta f(x,y;h,k)}{4hk} = L$ means for every $\epsilon > 0$ there is a $\delta > 0$ so that $0 < h,k < \delta$ implies $\frac{\Delta f(x,y;h,k)}{4hk} - L < \epsilon$. In Ash, Cohen, Freiling, and Rinne [1] is the following theorem.

Theorem ACFR: If $f(x,y)$ is a continuous function on R^2 and

$$\lim_{h,k\to 0} \frac{\Delta f(x,y;h,k)}{4hk} = 0$$

for all (x,y), then there are one-variable functions a and b so that $f(x,y) = a(x) + b(y)$.

One question this led to is what happens if the ratio of h and k is controlled somehow in this limit. Specifically, fix a positive number r, and suppose we only consider differences where $k \geq rh$, which we will indicate by $\Delta_r f(x,y;h,k)$. By using certain one-dimensional partitioning properties, we will obtain the conclusion of Theorem ACFR for the restricted derivative $\lim_{k\to 0} \frac{\Delta_r f(x,y;h,k)}{4hk}$. A simple example shows that these results do not hold for arbitrary functions. Let

$$f(x,y) = \begin{cases} 1 & x < y \\ 0 & x = y \\ -1 & x > y \end{cases}.$$