Real Analysis Exchange Vol.16 (1990–91)
S. Leader, Mathematics Department, Rutgers University, New Brunswick, NJ 08903.

Variation of f on E and Lebesgue Outer Measure of fE

Let f be a real-valued function on a cell $K = [a,b]$. By "cell" we mean a closed, bounded, nondegenerate interval in $\mathbf{R}$. The total variation of f is given by a Kurzweil-Henstock integral $\int_K |df| \leq \infty$ defined as the gauge-filtered limit of approximating sums over cell divisions with endpoint tags. For a development of this type of integral and its associated definition of differential, see [3,4,5]. We hope the reader will be impressed with the utility of our differential formulations based on an "honest" defintion of differential. We define the variation of f on a subset E of K to be the upper integral $\overline{\int}_K 1_E|df| \leq \infty$ where 1_E is the indicator of E. We call E df-null if this integral is zero, that is, if the differential $1_E\, df = 0$ [3,4]. Before the advent of the Kurzweil-Henstock integral df-null sets E were treated indirectly by using the condition that the image fE be Lebesgue-null. Indeed, as we shall show in Theorem 2, fE is Lebesgue-null if E is df-null. This result enables us to avoid the usual tedious proofs that an image fE is Lebesgue-null by resorting to a concise proof of the inherently stronger condition that E is df-null. Theorem 11 gives a converse to Theorem 2 for f a continuous function of bounded variation. For such f a set E is df-null if and only if fE is Lebesgue-null. So for continuous f of bounded variation Lusin's condition (N) that f map Lebesgue-null sets into Lebesgue-null sets is obviously just the absolute continuity conditon that every Lebesgue-null set is df-null. Let m be Lebesgue measure and m^* be Lebesgue outer measure.

Theorem 1. Let E be a subset of K such that at each point of E f is either left or right continuous. Then

$$m^*(fE) \leq 2\overline{\int}_K 1_E|df|. \tag{1}$$

Proof: Let D be the set of those t in E for which there exist cells J containing t with diam $fJ = 0$, that is, with f constant on J. Clearly fD is countable, so $m(fD) = 0$. Given a gauge δ on K and $\varepsilon > 0$ each t in $E \setminus D$ is an endpoint of some cell J in K such that (J,t) is δ-fine and $0 <$ diam $FJ < \varepsilon$. Given $c > 1$ choose s in J such that