Inroads *Real Analysis Exchange Vol.16 (1990–91)*
M. Bhaskara Rao, North Dakota State University, Fargo, ND 58105
K. Subramanyam, North Carolina State University, Wilmington, NC 28403

Measures With Prescribed Marginals, Extreme Points and Measure Preserving Transformations

Let $(X, \mathcal{A}, \lambda)$ and $(Y, \mathcal{B}, \nu)$ be two probability spaces. Let $M(\lambda, \nu)$ be the collection of all probability measures μ on the product σ-field $\mathcal{A} \times \mathcal{B}$ of $X \times Y$ such that the first and second marginals of μ are λ and ν, respectively, i.e., $\mu_1(A) = \mu(A \times Y) = \lambda(A)$ for every A in $\mathcal{A}$, and $\mu_2(B) = \mu(X \times B) = \nu(B)$ for every B in $\mathcal{B}$. The set $M(\lambda, \nu)$ is convex. The extreme points of this set have been characterized by Douglas (1964, Theorem 1, p.243) and Lindenstrauss (1965) when $X = Y$, $\mathcal{A} = \mathcal{B}$, $\lambda = \nu$ and the probability space has some additional structure. Let T be a measure preserving transformation from X to Y, i.e., T is a measurable transformation from X to Y, and $\lambda(T^{-1}B) = \nu(B)$ for every B in $\mathcal{B}$. We show that every such transformation gives an extreme point of $M(\lambda, \nu)$. The basic idea is to build a probability measure μ_T in $M(\lambda, \nu)$ sitting on the graph $G = \{(x, Tx);\ x \in X\}$ of T. But the graph G of T need not be available in the product σ-field $\mathcal{A} \times \mathcal{B}$. See Rao and Rao (1981, p.17) or Rao (1969). We overcome this difficulty by proceeding as follows and obtain a measure μ_T for which G is a thick set.

Let P_1 be the projection map from $X \times Y$ to X. We claim that the graph G has the property: for every E in $\mathcal{A} \times \mathcal{B}$, $P_1(E \cap G) \in \mathcal{A}$. For, let $\mathcal{E} = \{E \in \mathcal{A} \times \mathcal{B};\ P_1(E \cap G) \in \mathcal{A}\}$. One can show that $\mathcal{E}$ is closed under complementation and countable unions, and contains all measurable rectangle sets. Hence $\mathcal{E} = \mathcal{A} \times \mathcal{B}$. Define a set function μ_T on $\mathcal{A} \times \mathcal{B}$ by

$$\mu_T(E) = \lambda(P_1(E \cap G)) \text{ for } E \text{ in } \mathcal{A} \times \mathcal{B}.$$

<u>**THEOREM.**</u> μ_T is an extreme point of $M(\lambda, \nu)$.

<u>**Proof.**</u> It is easy to check that μ_T is a probability measure on $\mathcal{A} \times \mathcal{B}$. We now check that μ_T has the prescribed marginals. Let $A \in \mathcal{A}$. Then $\mu_1(A) = \mu_T(A \times Y) = \lambda[P_1((A \times Y) \cap G))] = \lambda(A \cap T^{-1}Y) = \lambda(A)$. Let $B \in \mathcal{B}$. Then