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DERIVATIVES AND THE CARATHÉODORY SUPERPOSITION

Let $\mathbf{R}$ be the set of reals. The density topology T_d ([1], [8], [10]) on $\mathbf{R}$ consists of all measurable subsets A of $\mathbf{R}$ such that, for every $x \in A$, x is a density point of A. Let $I \subset \mathbf{R}$ be an interval. A function $f : I \to \mathbf{R}$ is density continuous ([5], [6], [7]) if it is continuous as a map from (I, T_d) into (R, T_d).

A family $\mathcal{F}$ of maps of the topological space $(\mathbf{R}, T_d)$ into $\mathbf{R}$ (with the natural topology) is said to be T_d-equicontinuous at a point $x \in \mathbf{R}$ ([9], p. 188), if, given $\varepsilon > 0$, there is a neighborhood $V \in T_d$ of x such that $|f(u) - f(x)| < \varepsilon$ for each $u \in V$ and $f \in \mathcal{F}$. We say that $\mathcal{F}$ is T_d-equicontinuous on $\mathbf{R}$ if it is T_d-equicontinuous at each point.

In the paper [2] I proved the following theorem:

__Theorem 0.__ Suppose that $D \subset \mathbf{R}^2$ is a nonempty open set and $f : D \to \mathbf{R}$ is a locally bounded function such that all sections $f^y(t) = f(t, y)$ $(t, y \in \mathbf{R}$ and $(t, y) \in D)$ are derivatives and all sections $f_x(t) = f(x, t)$ $(x, t \in R$ and $(x, t) \in D)$ are equicontinuous. Then for every continuous function $g : I \to \mathbf{R}$ such that $(x, g(x)) \in D$ for $x \in I$ and I is an interval, the function $h(x) = f(x, g(x))$ is a derivative.

In this paper we approach the derivative structure of the function h in terms of density continuity.

__Theorem 1.__ Suppose that $D \subset \mathbf{R}^2$ is a nonempty open set and $f : D \to \mathbf{R}$ is a locally bounded function such that all sections f^y are derivatives and all sections f_x are T_d-equicontinuous. Then for every continuous and density continuous function $g : I \to \mathbf{R}$ such that I is an interval and $(x, g(x)) \in D$ for $x \in I$, the superposition $h(x) = f(x, g(x))$ is a derivative.

__Proof.__ First, we remark that the function h is measurable in the Lebesgue sense ([4]). We shall prove that h is a derivative at each point $x \in I$, i.e.