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A Global Implicit Function Theorem

1. In this paper, $f(x,y)$ is a function of two variables defined on an open subset U of R^2. Let $D_1f(D_2f)$ denote the partial derivative of f with respect to the 1-st place (2-nd place) variable. We let $D_1^+f(D_2^+f)$ denote the upper right Dini derivate of f with respect to the 1-st place (2-nd place) variable. Likewise $D_1^-f(D_2^-f)$ denotes the upper left Dini derivate of f with respect to the 1-st place (2-nd place) variable.

As in [C] we say that the function f on U is locally bounded at a point $(x,y) \in U$ if f is bounded in some neighborhood of (x,y). It follows that f is locally bounded at (x,y) if f is continuous at (x,y).

The standard result on implicit functions for functions of two variables [Ct] is:

Theorem 0. Let f be continuously differentiable on U and let D_2f never vanish on U. Then any point $(x_0,y_0) \in U$ lies in a segment $I = \{(x,y_0) : a < x < b\}$ for which there is a differentiable function g defined on I such that $g(x_0) = y_0$, and $f(x,g(x)) = f(x_0,y_0)$ for $(x,y_0) \in I$; moreover, $g' = -D_1f/D_2f$ for $(x,y_0) \in I$.

In the spirit of [C], we offer a global theorem in which boundedness replaces continuity of the derivatives,

Theorem 1. Let f be a continuous function on U and let $D_1^+f, D_1^-f, D_2^+f, D_2^-f$ be each $< \infty$. Let $D_2^+f > 0$, and let $D_1^+(D_2^+f)$ be locally bounded on U. Then almost every point $(x_0,y_0) \in U$ lies in a segment $I = \{(x,y_0) \in U : a < x < b\}$ for which there is a continuous function g defined on I such that $g(x_0) = y_0$ and $f(x,g(x)) = f(x_0,y_0)$ for $(x,y_0) \in I$; moreover, at almost every point of I (relative to I) the derivatives g', D_1f and D_2f exist and $g' = -D_1f/D_2f$.

We also provide a variation, easier to prove, that employs Baire category instead of Lebesgue measure. We say that a set is residual if its complement is a first category set.

Proposition 1. Let f be a continuous function on U and let D_1f and D_2f exist on U. Let D_2f never vanish on U. Then there is a residual subset Z of U such that every point $(x_0,y_0) \in Z$ lies in a segment $I = \{(x,y_0) : a < x < b\}$