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Martin's Axiom implies a stronger version of Blumberg's Theorem

Let **R** be the real line. In 1922, H. Blumberg proved the following theorem: **Blumberg's Theorem** [Bl]: If $f:\mathbf{R}\to\mathbf{R}$, then there is a dense subset D of **R** such that $f|D$ is continuous. Here, $f|D$ is the real valued function on D with the subspace topology.

In any such theorem, it is of interest to ask how much the hypothesis can be weakened or the conclusion strengthened. The obvious way to weaken the hypothesis is to allow the domain of f to be some subset of **R** instead of **R**. A set $X\subseteq Y$ is **categorically dense** in Y if $X\cap U$ is of second category in Y for every nonempty open subset U of Y. Trivial modifications in the proof of Blumberg's Theorem then give the following strengthening:

Proposition: If X is a categorically dense subset of **R**, and $f:X\to\mathbf{R}$, then there is a dense $D\subseteq X$ such that $f|d$ is continuous.

If every point of $X\subseteq\mathbf{R}$ is isolated, then the same result holds trivially. For similar reasons, this is also true if X is scattered (just let D be the set of isolated points of X). However, if X is dense, it is easy to see that the hypothesis cannot be weakened any further, for if $X\subseteq\mathbf{R}$ is dense and of first category, partition X into countably many sets X_n, $n<\omega$, each nowhere dense. Let $f(x)=n$ iff $x\in X_n$, and f obviously cannot be continuous on any dense subset. If X is dense and of second category, but not categorically dense, the same trick can be used on $X\cap I$ for some interval I, letting f be constant outside I.

Thus, for dense X, X being categorically dense is both necessary and sufficient (at least for subsets of **R**). It is perhaps somewhat surprising that the