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ON APPROXIMATE DIFFERENTIABILITY AND BILATERAL KNOT POINTS

It is well known that for a measurable function f defined on the interval $[0,1]$ we have $D_-f = ADf = D^+f$ a.e. in $\{D^+f < +\infty\}$. (See [1], Theorem 2.) At the same time there are trivial examples of measurable functions f for which the approximate derivative ADf exists a.e., but almost every point $x \in [0,1]$ is a bilateral knot point, i.e. $D^+f(x) = D^-f(x) = +\infty$ and $D_+f(x) = D_-f(x) = -\infty$. In this note, answering a problem of K. M. Garg ([2], 10.3 Problem), we construct a continuous function with these properties.

Let us consider three sequences $\{c_n\}$, $\{d_n\}$, and $\{r_n\}$ of positive real numbers such that $\{d_n\}$ is decreasing, $c_n < d_n/2$, and

$$\lim_{n\to\infty} r_n d_n^{-1} = \infty , \tag{1}$$

$$\Sigma_{n=1}^{\infty} c_n d_n^{-1} < \infty , \tag{2}$$

$$\Sigma_{n=1}^{N-1} r_n c_n^{-1} \leq r_N/4d_N \quad \text{for each} \quad N \geq 2 , \tag{3}$$

$$\Sigma_{n=N+1}^{\infty} r_n \leq r_N/4 \quad \text{for each} \quad N \geq 1 . \tag{4}$$

In order to provide some examples of such sequences let us take a sequence $\{u_n\}$ of numbers from the interval $(0,1/2)$ satisfying $\Sigma_{n=1}^{\infty} u_n < \infty$, and let $\{w_n\}$ be an arbitrary sequence of positive real numbers which decreases to zero and satisfies the inequality $\Sigma_{n=1}^{N-1} (u_n w_n)^{-1} \leq (4w_N)^{-1}$ for every $N \geq 2$. Next, for arbitrary fixed number $p \geq 5$, put

$$r_n = p^{-n} , \quad d_n = w_n r_n , \quad c_n = u_n d_n \quad (n \geq 1) .$$