Real Analysis Exchange
Vol. 19(2), 1993/94, pp. 660–662

Michael Schramm, Department of Mathematics, LeMoyne College, Syracuse, New York 13214, e-mail: schramm@maple.lemoyne.edu

ON HBV AND THE GARSIA–SAWYER CLASS

We consider continuous functions with domain $[a, b]$ and range $[c, d]$. Such a function f is said to be in the Garsia–Sawyer Class (GS) if $\int_c^d \log^+(n_f(y))\, dy$ is finite, where $n_f(y)$ is the Banach indicatrix of f. Garsia and Sawyer [2] showed that functions in GS have uniformly convergent Fourier series. Let $\Phi = \{\varphi_n\}$ be a sequence of convex functions with the following properties:

i) $\varphi_n : [0, \infty) \to [0, \infty)$ for $n = 1, 2, \ldots;$

ii) $\varphi_n(0) = 0$ and $\varphi_n(x) > 0$ for $x > 0,\ n = 1, 2, \ldots;$

iii) $\varphi_{n+1}(x) \le \varphi_n(x)$ for $x \ge 0,\ n = 1, 2, \ldots;$

iv) $\sum_{n=1}^{\infty} \varphi_n(x) = \infty$ for $x > 0.$

We have said [3] that f is of Φ-Bounded Variation (ΦBV) if there is a positive constant c so that $\sum \varphi_n\left(c\,|f(b_n) - f(a_n)|\right)$ is finite for any collection $\{[a_n, b_n]\}$ of non-overlapping subintervals of $[a, b]$ (and this is equivalent to requiring such sums to be uniformly bounded). By making appropriate choices of the functions φ_n, we may obtain many of the spaces of generalized bounded variation that have been studied. In particular, if $\varphi_n(x) = x/n$, we have the functions of Harmonic Bounded Variation (HBV), introduced by Waterman [4] (in this case we may take $c = 1$ above). Waterman showed that continuous functions in HBV have uniformly convergent Fourier series, and moreover [5] that GS $\subsetneq$ HBV. HBV is pivotal in this context, since if ΦBV properly contains HBV, there is a continuous function in ΦBV whose Fourier series diverges at a point. But GS is not closed under addition, so GS is not the same as HBV (an illustration of this fact may be found in [1]). The full story of the relationship between GS and HBV is not yet known. Here we establish a result relating to the way GS is distributed through HBV.

Key Words: Garsia–Sawyer, bounded variation, harmonic bounded variation
Mathematical Reviews subject classification: Primary 26A45
Received by the editors February 16,1994