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ON TAU-SMOOTH MEASURE SPACES WITHOUT THICK LINDELÖF SUBSETS

1. Introduction. Let X be a completely regular Hausdorff space. The smallest σ–algebra of subsets of X making all the real valued continuous functions on X measurable (or, equivalently, the σ–algebra generated by the cozero sets of X) is denoted by $Ba(X)$, the Baire sets of X; and the σ–algebra generated by the open sets of X is denoted by $Bo(X)$, the Borel sets of X.

We say that a collection $\mathcal{N}$ of subsets of X is directed upwards if whenever $O_1, O_2 \in \mathcal{N}$, there exists a set $O_3 \in \mathcal{N}$ with $O_1 \cup O_2 \subset O_3$. A countably additive measure μ defined on $Ba(X)$ is said to be *τ–smooth* or *τ–additive*, if for any upwards directed collection $\mathcal{N}$ of cozero subsets of X such that $\cup \mathcal{N} = X$, we have $\sup_{O \in \mathcal{N}} \mu(O) = \mu(X)$. A Borel measure μ is *τ–smooth* if for any upwards directed collection $\mathcal{N}$ of open subsets of X, we have $\sup_{O \in \mathcal{N}} \mu(O) = \mu(\cup \mathcal{N})$.

In [1] Robert F. Wheeler asks the following question (Problem 8.14): Is it true that if μ is a finite τ–smooth Baire measure on X, then there is a Lindelöf subset of X with full μ–outer measure? For the sake of brevity we say that a τ–smooth measure space $(X, Ba(X), \mu)$ has the L property if there is a μ–thick Lindelöf subset of X. A set $B \subset X$ is thick (or equivalently, has full outer measure) if the inner measure of the complement is zero. Clearly, a measure space with the L property is τ–smooth, since every cover of X by cozero sets contains a countable subcollection whose union has full measure; so Wheeler's question amounts to asking whether the L property characterizes the measure spaces that are τ–smooth. In this note we show that a negative answer is consistent with **ZFC**. We investigate the specific case of the Sorgenfrey plane with Lebesgue measure, and prove that whether it has the L property or not is undecidable in **ZF** (assuming inaccesible cardinals exist). More precisely, it will be shown that

i) there is a model of **ZF** where the Sorgenfrey plane with Lebesgue measure lacks the L property;

ii) under **ZFC + CH** the Sorgenfrey plane with Lebesgue measure has the L property;