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QUASI-COMPONENTS OF PREIMAGES OF A CONNECTIVITY FUNCTION $I^2 \rightarrow I$

Very little is known about what properties are satisfied by the composite of any two connectivity functions. In 1959, Stallings [6] asked under what circumstances will the composition of almost continuous functions be almost continuous? He indicated how to construct an example of connectivity functions $I^2 \rightarrow I$ and $I \rightarrow I$ whose composite fails to be a connectivity function $I^2 \rightarrow I$, where $I = [0,1]$. In 1973, Kellum [3] gave an example of almost continuous functions $f : I^n \rightarrow I^m$ and $g : I^m \rightarrow I^n$ so that $g \circ f : I^n \rightarrow I^n$ has no fixed point and is not almost continuous. When $n = m = 1, f$ and g are then connectivity functions [6].

It is well known that if a function $h : X \rightarrow Y$ is the composite of connectivity functions $f : X \rightarrow Y$ and $g : Y \rightarrow Y$, then h must be a Darboux function. In this note we give a simple example of an almost continuous Darboux function $h : I^2 \rightarrow I$ for which the converse is false. To verify this example, we rely on either of two useful results about quasi-components. Kellum's question about whether the converse is true when $X = Y = I$ is still unanswered. We also give a sufficient condition on quasi-components in order for a function $I^2 \rightarrow I$ to be Darboux.

A function $f : X \rightarrow Y$ is defined to be a <u>Darboux</u> (<u>connectivity</u>) function if $f(C)$ (the graph of the restriction $f|C$) is connected for every connected subset C of X. We say $f : X \rightarrow Y$ is <u>almost</u> <u>continuous</u> if each open neighborhood of the graph of f in $X \times Y$ contains the graph of a continuous function from X into Y. A function $f : X \rightarrow Y$ is <u>peripherally</u> <u>continuous</u> if for each $x \in X$ and each open neighborhood U of x and V of $f(x)$, there is an open neighborhood W of x in U such that $f(bd(W)) \subset V$. According to [6], if $X = I^n$ and $n \geq 2$, then W and $bd(W)$ can be chosen to be connected. For functions $f : I^n \rightarrow I^m$, $n \geq 2$, we have: peripheral continuity $\Leftrightarrow$ connectivity $\Rightarrow$ almost continuity [2], [7], [6].

A set $A \subset X$ has <u>external</u> <u>dimension</u> 0 if for every $p \in X - A$, each open neighborhood of p contains an open set about p whose boundary misses A. If $Q \subset B \subset X$, we say Q is a <u>quasi-component</u> of B provided Q is a maximal set which lies in one of two separated sets D or E whenever $B = D \cup E$.

The following result comes out of Whyburn's work [8].