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ON THE MEASURABILITY PROBLEM FOR CATEGORY BASES

In an effort at unifying properties which are common for measure and category, John C. Morgan II defined the concept of a category base. Namely, a family $\mathbb{C}$ is said to be a *category base* on X if $\mathbb{C}$ is a family of non-empty subsets of X, called *regions*, satisfying the following axioms: (1) $\cup\mathbb{C} = X$, (2) if $\mathcal{S} \subseteq \mathbb{C}$ and $A \in \mathbb{C}$ are such that $|\mathcal{S}| < |\mathbb{C}|$, $\mathcal{S}$ is pairwise disjoint and $A \cap B$ contains no region for each $B \in \mathcal{S}$, then there exists a region D such that $D \subseteq A - (\cup\mathcal{S})$. Although the definition given here differs from the original one (cf. [M1]), they are trivially equivalent. With respect to a given category base $\mathbb{C}$ on X, one can classify the subsets of X in the following way. A set $E \subseteq X$ is *singular* if each region contains a subregion disjoint from E. A set $M \subset X$ is *meager* if it is a union of countably many singular sets; the family of all meager sets for $\mathbb{C}$ is denoted by $\mathcal{M}(\mathbb{C})$. A set $B \subset X$ is *Baire* if every region contains a subregion A such that either $A \cap B$ or $A - B$ is meager; the family of all Baire sets for $\mathbb{C}$ is denoted by $\mathcal{B}(\mathbb{C})$.

A typical example of a category base is the family of measurable sets of positive measure with respect to a σ-finite complete measure. In this case, or even in the case of a *ccc* complete measure space, the meager sets coincide with the sets of measure zero and the Baire sets with measurable ones (see [M3]). K. Schilling [Sc] showed that there exists a complete and non-atomic measure space such that the family of all measurable sets is not the set of all Baire sets for any category base. In connection with this the following Measurability Problem was posed by J.C. Morgan II (cf [M2; p.379]): Characterize those complete measure structures for which the measurable sets coincide with the Baire sets for a suitable category base of measurable sets. We provide two such characterizations (Theorem 3 and Theorem 4) but with an additional stipulation that the measure zero sets coincide with the meager sets. The lack of this stipulation may yield measurable cardinals (see Proposition 2).

Our basic sources for related results and for used but undefined concepts are as follows: set-theoretical – Jech's book [J], measure-theoretical – Fremlin's survey paper [F] and category base theoretical – Morgan's book [M1].