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A REMARK ON ABSOLUTELY CONTINUOUS FUNCTIONS

In a previous note, [1], the authors obtained the following Lusin type theorem for absolutely continuous functions.

THEOREM 1. *A real function f on $[0,1]$ is absolutely continuous if and only if for every $\varepsilon > 0$ there is a g which is continuously differentiable such that the set E for which $f(x) \neq g(x)$ has measure less than ε and $\int_E |f'(t)|dt < \varepsilon$, $\int_E |g'(t)|dt < \varepsilon$.*

Our present purpose is to show that this result is, in a certain sense, best possible. In this regard, we need a known fact, [2], about continuous functions, to be described presently. Let C be the set of continuous functions on $[0,1]$ and, for every modulus of continuity w, let C_w be the functions in C which satisfy this modulus.

THEOREM 2. *For every modulus of continuity w, there is an $f \in C$ such that; for every $g \in C_w, f(x) \neq g(x)$ almost everywhere.*

We now state and prove the fact we need, which follows from Theorem 2.

THEOREM 3. *For every modulus of continuity w, there is an $f \in C^1$ such that, for every $g \in C^1$ with $g' \in C_w, f(x) \neq g(x)$ almost everywhere.*

Proof. Let $h \in C$ be such that, for every $k \in C_w, h(x) \neq k(x)$ almost everywhere. Let $f(x) = \int_0^x h(t)dt, 0 \leq x \leq 1$. Suppose there is a differentiable g such that $g' \in C_w$ and $f(x) = g(x)$ on a set E for which $m(E) > 0$. Almost every point of E is a point of metric density 1 of E. Clearly, for every such x, $f'(x) = g'(x)$. Hence, the set of points for which $f'(x) = g'(x)$ has positive measure. But, $f'(x) = h(x)$, for every x. Since $g' \in C_w$, this contradicts the assumed property of h.

COROLLARY 1. *For every modulus of continuity w there is an absolutely continuous f such that, for every $g \in C^1$, with $g' \in C_w, f(x) \neq g(x)$ almost everywhere.*

We remark that in connection with this topic the following metric suggests itself for the set of absolutely continuous functions. For f and g in AC, let

$$d(f,g) = m(E) + \int_E |f'(t)|dt + \int_E |g'(t)|dt,$$

where E is the set for which $f(x) \neq g(x)$. It is a fact that AC is a Banach space with this metric.