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A CHARACTERIZATION OF NON-ATOMIC PROBABILITIES ON [0,1] WITH NOWHERE DENSE SUPPORTS

For a countably additive Borel probability measure μ on $[0,1]$, let $\{T_i(\mu) : i \in N\}$ be an enumeration of the connected components of $[0,1]\backslash\text{supp}(\mu)$. These are the intervals of constancy of the cumulative distribution function F_μ. For all i let $y_i(\mu)$ be the value of F_μ on $T_i(\mu)$.

Proposition 1 *μ is non-atomic with supp(μ) nowhere dense iff $\{y_i(\mu) : i \in N\}$ is dense in* $[0,1]$.

Proof: Suppose that μ is non-atomic with nowhere dense support. Since μ is non-atomic F_μ is continuous and $\{F_\mu(x) : x \in \text{supp}(\mu)\} = [0,1]$. If $0 \leq y_1 < y_2 \leq 1$ are $F_\mu(x_1)$ and $F_\mu(x_2)$ with $x_1 < x_2$ in supp(μ) there is an interval $T_i(\mu)$ between x_1 and x_2 since supp(μ) is nowhere dense. Thus $y_1 < y_i(\mu) < y_2$. This establishes density of $\{y_i(\mu) : i \in N\}$ in $[0,1]$.

Assume density of $\{y_i(\mu) : i \in N\}$. μ must be non-atomic for otherwise there would be an $x \in [0,1]$ so that $F_\mu(x^-) = \lim_{z\uparrow x} F_\mu(z) < F_\mu(x)$. In this case no $y_i(\mu)$ would be in $(F_\mu(x^-), F_\mu(x))$ contradicting density. supp(μ) must be nowhere dense for if $\phi \neq (x_1, x_2) \subset \text{supp}(\mu)$ then $F_\mu(x_1) < F_\mu(x_2)$ so $y_i(\mu) \in (F_\mu(x_1), F_\mu(x_2))$ for some $i \in N$ hence $T_i(\mu)$ is in (x_1, x_2) which is impossible since $T_i(u) \cap \text{supp}(u) = \phi$. Thus supp($\mu$) is nowhere dense.

□

The intervals $\{T_i(\mu) : i \in N\}$ are non-overlapping and are ordered by $T_i(\mu) < T_j(\mu)$ iff $x_i \in T_i(\mu)$ and x_j in $T_j(\mu)$ implies $x_i < x_j$. The mapping $y_i \rightarrow T_i(\mu)$ is an order isomorphism. $\{y_i(\mu) : i \in N\}$ has maximum 1 (minimum 0) iff $\{T_i(\mu) : i \in N\}$ has a maximum containing 1 (minimum containing 0) iff $1 \notin \text{supp}(\mu)$ ($0 \notin \text{supp}(\mu)$). Allowing for different possible order types the converse is true. If K is a perfect nowhere dense subset of $[0,1]$ and the countable dense subset $\{y_i : i \in N\}$ of $[0,1]$ has extrema of the same type as the components $\{T_i : i \in N\}$ of $[0,1]\backslash K$ there is an order isomorphism $T_i \leftrightarrow y_i$ (see Theorem 1 page 160 of Fraenkel [1961]). For such an isomorphism define $F(x) = y_i$ if $x \in T_i$ to obtain a non-decreasing function from $[0,1]\backslash N \rightarrow [0,1]$ which has a right continuous extension (which is continuous

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