Real Analysis Exchange Vol 15 (1989-90)

Li Baoling, Department of Mathematics, Northwest Normal University, Lanzhou, China

CONVERGENCE THEOREMS FOR THE VARIATIONAL INTEGRAL

1. Introduction

The variational integral is a kind of nonabsolute integrals originally defined by R. Henstock [1]. It is equivalent to the Riemann complete integral [2]. In [3], Yoto Kubota has shown some elementary properties of the integral, including the important Cauchy and Harnack extensions. In this paper, we shall establish some significant convergence theorems for the integral.

Definition 1.1 Let $[a,b]$ be a compact interval on the real line, and $\delta(\xi)$ a positive real function defined on $[a,b]$. The finite set

$$P = \{x_0, x_1, \ldots, x_p; \xi_1, \ldots, \xi_p\} \tag{1.1}$$

is said to be a δ-fine division over $[a,b]$ if

$$a = x_0 < x_1 < \ldots < x_p = b \text{ and}$$
$$\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \text{ for } \quad i = 1, 2, \ldots, p.$$

Alternatively, we write

$$P = \{[u,v]; \xi\} \tag{1.2}$$

where $[u,v]$ denotes a typical subinterval in the division, and

$$\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi)). \tag{1.3}$$

Definition 1.2 An interval function S is said to be superadditive if, for any two adjacent non-overlapping intervals I_1 and I_2, we have

$$S(I_1 \cup I_2) \geq S(I_1) + S(I_2) \tag{1.4}$$

Definition 1.3 Let $f : [a,b] \longrightarrow R$ be a measurable function. Then f is said to be variationally integrable on $[a,b]$ if there is a function F such that, for every $\epsilon > 0$, there is a $\delta(\xi) : [a,b] \longrightarrow (0,\infty)$, and a superadditive interval function S such that

$$0 = S([a,a]) \leq S([a,b]) < \epsilon \tag{1.5}$$

and that whenever $\xi - \delta(\xi) < u \leq \xi \leq v < \xi + \delta(\xi)$ we have