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Restriction and Intersection Theorems in Real Analysis

This talk is about my Ph. D. thesis [Bu7] written in Hungarian in 1987. Although all of the theorems and proofs of this thesis are available in English [Bu1-6] I think that the abstract of this thesis might be interesting to an audience bigger than those who can read Hungarian.

The study of the level set of functions is among the basic tools of Real Analysis. A nice example for this method is due to N. Bary and D. Menchoff (1928, 1930) [S, Ch. IX, 8.]: in order that a continuous function f be representable as a superposition of two absolutely continuous functions, it is necessary and sufficient that almost every one of its values is assumed at most a finite number of times and $|f(H)| = 0$ if $|H| = 0$ where $|H|$ denotes the Lebesgue measure of H. Another result is due to E. Čech [Ce] (1931). He proved that if f is continuous on an interval and its every level set is finite then f is monotone on a subinterval.

Obviously the level set can be regarded as the projection to the $x-$axis of the intersection of the graphs of f and a constant function. The study of the intersection properties of wider function classes is a generalization of the level set technique. Here are some possible questions:

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are sets of functions.

(i) If the functions in $\mathcal{A}$ intersect every function in $\mathcal{B}$ in "nice" sets then what regularity properties will we have for the functions in $\mathcal{A}$?

(ii) Find properties $\mathcal{P}$ such that for every $f \in \mathcal{A}$ we can find a $g \in \mathcal{B}$ such that the intersection of the graphs of f and g has property $\mathcal{P}$.

(iii) For a given property $\mathcal{P}$ find some "pathological" functions in $\mathcal{A}$ such that for every $g \in \mathcal{B}$ the intersection of f and g has property $\mathcal{P}$.

The theorems of Bary-Menchoff and Čech are examples for the questions of type (i). An example for type (ii) is the following theorem of Filipczak [F] (1966): For every continuous function f there exists a non-empty perfect set P such that $f|_P$, the restriction of f to P, is monotone. Thus for every continuous function f there exists a monotone function g such that the intersection set of f and g contains a perfect set. This example shows why we speak about restriction and intersection theorems. In most of the cases if we can prove that the restriction of a function in $\mathcal{A}$ has "nice" properties, then using an extension theorem one can obtain an intersection theorem from the restriction theorem.

An example for type (iii) is due to Gillis [B, Ch. XIII. 3.]: There exists a continuous non-constant function such that its every level set is perfect.

A. Bruckner, J. Ceder and M. Weiss [BCW] (1969) and M. Laczkovich [L] (1984) have several results about differentiable restrictions of continuous functions. Namely, if f is defined on a non-empty perfect set P then there exists a non-empty $P' \subset P$ such that the restriction of f onto P' has a finite or infinite derivative everywhere. If $|P| > 0$ then $f|_{P'}$ can be differentiable.