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HEARTS DENSITY THEOREMS

The Lebesgue density theorem is clearly one of the most important results of real analysis. On the other hand, from the general point of view it just claims that one out of the many differentiation systems on the real line has the density property. However, since the Lebesgue density theorem (LDT) plays the key role in so many questions of real analysis, one might expect that it is canonical in some sense, e.g. that it is the only density theorem connected in a reasonable way with the algebraic structure of the reals. In the language of abstract density topologies this question was asked by L. Zajíček at Scuola di Analisi Reale, Ravello 1985 and in [Z].

Here we intend, after giving a precise formulation of the problem, to point out several other density theorems connected with the algebraic structure of the reals. This shows, surprisingly enough, that the LDT is not canonical in the sense of translation invariance nor in the sense of affine invariance.

Let us recall that in a general differentiation system (DS) (for the Lebesgue measure (denoted by $|.|$) on the real line R) to each $x \in R$ there correspond Moore-Smith sequences (families filtering to the right) of sets of finite and positive measure. (Cf. [HP].) A point $x \in R$ is said to be a density point of a measurable set $E \subset R$ (for the given DS) if

$$\lim |E \cap E_t| / |E_t| = 1$$

for every Moore-Smith sequence corresponding to x.

In an obvious way one defines the notion of the density property (DP): A DS has the DP if almost every point of every measurable set is its density point.

For example, in the Lebesgue differentiation system (LDS) to each $x \in R$ there corresponds the Moore-Smith sequence of intervals $(x-h, x+h)$ (with $h \searrow 0$). In this language the LDT says that the LDS has the DP.

For any DS Θ one of the following situations may happen: