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A MOMENT INEQUALITY

1. Introduction.

In his doctoral thesis, H. Thunsdorff proved the following inequality.

Theorem [T]. If $f:[0,1] \to \mathbb{R}$ is a nonnegative, convex function such that $f(0) = 0$ and $0 < m \leqslant n < +\infty$, then

$$(1) \qquad \left[(m+1)\int_0^1 f^m\,dx\right]^{1/m} \leqslant \left[(n+1)\int_0^1 f^n\,dx\right]^{1/n}.$$

(See [NS] for an elementary proof of this inequality.)

It was pointed out in [N] that the classical inequality

$$(2) \qquad \left[\int_0^1 f^m\,dx\right]^{1/m} \leqslant \left[\int_0^1 f^n\,dx\right]^{1/n},$$

for nonnegative, measurable functions $f:[0,1] \to \mathbb{R}$ and $0 < m \leqslant n < +\infty$, implies the inequality

$$(3) \qquad \left[(m+1)\int_0^1 f^m\,dx\right]^{1/m} \leqslant e\left[(n+1)\int_0^1 f^n\,dx\right]^{1/n},$$

where the constant e is sharp even for the subclass of nondecreasing function $f:[0,1] \to \mathbb{R}$. In the same paper [N], a class of nondecreasing functions for which the inequality (1) holds was investigated. We give the theorem below for completeness sake.

Theorem [N]. Let $f:[0,1] \to \mathbb{R}$ be a nondecreasing function with $f(0) = 0$. If the closure of the planar set $\{(x,y) \mid f(x) \leqslant y$ and $x \in [0,1]\}$ is star-like with respect to the origin $(0,0)$ and $0 < m \leqslant n < +\infty$, then we have that the inequality (1) holds true.