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A RIESZ-TYPE DEFINITION OF THE DENJOY INTEGRAL

Riesz [4] defines a Lebesgue integrable function as the almost everywhere limit of a mean convergent sequence of step functions. A short proof of the uniqueness of the definition can be found in [2]. In this note we give a similar definition for the Denjoy integral and show that using this definition a convergence theorem can be proved.

First, we give some definitions [6]. Let X be a closed set in [a,b]. A function F is said to be absolutely continuous in the restricted sense on X or $AC_*(X)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever

$$\sum_i |b_i - a_i| < \delta$$

where $[a_i, b_i]$, $i = 1,2,\ldots$, is a finite or infinite sequence of nonoverlapping intervals in [a,b] and $a_i, b_i \in X$ for all i, we have

$$\sum_i \omega(F; [a_i, b_i]) < \varepsilon$$

where ω denotes the oscillation of F over $[a_i, b_i]$. Then F is ACG_* if [a,b] is the union of closed sets X_i, $i = 1,2,\ldots$, such that F is $AC_*(X_i)$ for each i. A function f is *Denjoy integrable* on [a,b] if there exists a continuous and ACG_* function F such that the derivative $F'(x) = f(x)$ almost everywhere in [a,b].