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DERIVATIVES ON COUNTABLE DENSE SUBSETS

Throughout this paper, A, B, C and D will be mutually disjoint countable dense subsets of the real line, R. In [1], p. 72, A. Bruckner observed that there exists an everywhere differentiable function f that has a proper local maximum at each point of A and at no other points, and has a proper local minimum at each point of B and at no other points. We will prove (Lemma 1) the existence of an increasing differentiable homeomorphism of R onto R that maps certain countable subsets of R the way we please. (Compare with [4], and also note the Remark on p. 72 of [1].) We use Lemma 1, together with one prototype function from [1], to prove:

Theorem 1. There is an everywhere differentiable function F on R that has a proper local maximum at each point of A and at no other other points, has a proper local minimum at each point of B and at no other points, and is increasing at each point of C and decreasing at each point of D.

Of course F must increase (decrease) at uncountably many points, so C(D) cannot include all such points. We also use Lemma 1 to construct a variety of types of pathological functions. The first type we consider are continuous functions that have no derivative, finite or infinite, at any point.

Theorem 2. There is a continuous function F on R such that at each $x \in R$, we have $[D_+F(x), D^+F(x)] \cup [D_-F(x), D^-F(x)] = [-\infty,\infty]$, where D^+, D_+, D^-, D_- denote the usual Dini derivates, and such that

$$D_+F(x) = \infty \quad \text{for} \quad x \in A, \quad D^+F(x) = -\infty \quad \text{for} \quad x \in B,$$

$$D_-F(x) = \infty \quad \text{for} \quad x \in C, \quad D^-F(x) = -\infty \quad \text{for} \quad x \in D.$$

By a <u>knot</u> point of f, we mean an $x \in R$ at which

$$D^+f(x) = D^-f(x) = \infty, \quad D_+f(x) = D_-f(x) = -\infty.$$