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Russell A. Gordon, Department of Mathematics, Whitman College, Walla Walla, WA 99362

On the Equivalence of two Convergence Theorems for the Henstock Integral

There are two well-known convergence theorems for the Henstock integral. Let $\{f_n\}$ be a sequence of Henstock integrable functions that converge pointwise to a function f on $[a,b]$ and let $F_n(x) = \int_a^x f_n$ for each n. One of the theorems has an easy proof and requires that the sequence $\{f_n\}$ be uniformly Henstock integrable on $[a,b]$. The other, usually written in the language of the Denjoy-Perron integral, has a more difficult proof and requires that the sequence $\{F_n\}$ be equicontinuous and equi ACG_* on $[a,b]$. It is not easy to compare the hypotheses of these two theorems. Two recent attempts ([1] and [2]) have been made. In [2], the term equi ACG^∇ is introduced. It is shown that $\{F_n\}$ is equi ACG^∇ on $[a,b]$ if and only if $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$. Furthermore, it is possible to prove that $\{F_n\}$ equi ACG_* on $[a,b]$ implies $\{F_n\}$ equi ACG^∇ on $[a,b]$. In [1], a new convergence theorem is proved and it is shown to include both of the standard convergence theorems as special cases. Here the necessary hypothesis is that $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on $[a,b]$. The purpose of this paper is to prove that $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$ if and only if $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on $[a,b]$.

We will assume that the reader is familiar with the terminology of the Henstock integral. The relevant notation needed for the paper appears below. Let $f, F : [a,b] \to R$, let $E \subset [a,b]$, let δ be a positive function defined on $[a,b]$, and let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \le i \le q\}$ be a finite collection of non-overlapping tagged intervals in $[a,b]$. Then

$f(\mathcal{P}) = \sum_{i=1}^{q} f(x_i)(d_i - c_i)$ denotes the Riemann sum of f associated with $\mathcal{P}$;

$F(\mathcal{P}) = \sum_{i=1}^{q} \big(F(d_i) - F(c_i)\big)$, where F will always be an indefinite integral;

CE denotes the complement of E;

$\rho(x, E)$ denotes the distance from x to E; and

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