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Robert E. Zink, Department of Mathematics, Purdue University, West Lafeyette, IN, 47907-1395

SCHAUDER BASES, SCHAUDER FUNCTIONS, AND THE GRAM-SCHMIDT PROCESS

A linearly independent subset of a vector space is a basis for the space, if every vector can be expressed as a linear combination of the basis vectors. In the finite–dimensional case, each basis contains the same (finite) number of elements, and the space coincides with the span of any one of these finite subsets, a property not enjoyed, however, by any infinite–dimensional space, $L^2[0,1]$ being a case in point. One possible modification of this concept of basis would allow the linear combinations to involve denumerably many vectors, but then, of course, one would need some notion of convergence for the series thus arising. This, in turn, leads one to conduct the discussion within the friendly confines of a distinguished class of vector spaces.

Let X be a Banach space; i.e., a complete, normed linear space, or, in other words, a normed vector space that is (Cauchy) complete in the topology engendered by its norm. A denumerable subset of X, $\{x_n : n = 1,2,\ldots\}$, is a Schauder basis for X iff each x in X has a unique representation $x = \sum_{n=1}^{\infty} c_n x_n$, in the sense that $\lim_n \|x - \sum_{i=1}^{n} c_i x_i\| = 0$. Thus, by virtue of the Riesz–Fischer theorem, every complete orthonormal system associated with an interval $[a,b]$ is a Schauder basis for $L^2[a,b]$. Familiar examples of this type are the trigonometric system, for $L^2[-\pi,\pi]$; the Legendre system, for $L^2[-1,1]$; and the Haar system, for $L^2[0,1]$.

The name for this sort of basis derives from an article written by Schauder [9] in which the concept is defined, and, among other things, it is shown how to construct bases for $C[0,1]$, the space of all real–valued, continuous functions on $[0,1]$, a Banach space when endowed with the supremum norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}.$$

Key Words: Schauder basis, Schauder series, Haar system

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