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THE FRÉCHET BOUNDS REVISITED

What have become known as the Fréchet bounds,

$$\max(F(x)+G(y)-1,0) \leq H(x,y) \leq \min(F(x),G(y)),$$

were published by Fréchet [2] in 1951 and even earlier by Hoeffding [3] in 1940. Here H is the joint distribution function of a pair X, Y of random variables whose one-dimensional distribution functions are F and G, respectively. It is well-known that $H(x,y)$ is identically equal to its Fréchet upper (lower) bound if and only if the mass of H is concentrated on a nondecreasing (nonincreasing) curve. Fréchet (1951) discussed this result in both the discrete case and the continuous case, and went on to say that things worked in essentially the same way in the general case. Hoeffding (1940) discussed the continuous case and said that his discussion of the discontinuous case would appear elsewhere. Motivated to some extent, perhaps, by the relative inaccessibility of these papers but also, undoubtedly, by a desire for a "better" proof, a number of others, including Dall'Aglio [1], Kimeldorf and Sampson [4], and Wolff [5], have since given proofs.

To our knowledge, each proof in the literature is either limited to the discrete or continuous case or else is quite sketchy. Perhaps it is fair to say that the literature even lacks a clear formulation of the result. Our purpose, in this paper, is to give a clear formulation of the result accompanied by a simple proof which makes no such assumptions about the nature of the marginals, F and G.

We begin with a definition. A subset S of $\mathbb{R}^2$ is *nondecreasing* if and only if, for all $(x,y),(u,v)$ in S,

$$x < u \text{ implies } y \leq v.$$

Lemma 1 *Let $S \subset \mathbb{R}^2$ be nondecreasing. Let (x,y) be an arbitrary element of $\mathbb{R}^2$. Either*

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