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ON THE DARBOUX PROPERTY OF THE SUM OF CLIQUISH FUNCTIONS

Let $\mathbf{R}$ be the set of reals. A function $f : \mathbf{R} \to \mathbf{R}$ is said to be cliquish at a point $x \in \mathbf{R}$ ([1]) if for every $\varepsilon > 0$ and for every open neighborhood U of x there exists a nonempty open set $V \subset U$ such that $\operatorname{osc}_V f \le \varepsilon$. Observe that $f : \mathbf{R} \to \mathbf{R}$ is cliquish at each point $x \in \mathbf{R}$ iff the set of its continuity points is dense.

In 1987 , H. W. Pu and H. H. Pu established the following theorem (See [2].):

Theorem P.P. *Let A be a finite family of Baire 1 functions. Then there exists a Baire 1 function f such that $f + g$ is a Darboux function for every $g \in A$.*

In this paper I prove that this theorem is true for finite families A of cliquish functions.

Let $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$. For a given function $f : \mathbf{R} \to \overline{\mathbf{R}}$ such that the set $\{x \in \mathbf{R} : f(x) = +\infty \text{ or } -\infty\}$ is nowhere dense, let $C(f)$ be the set of continuity points of f and let $D_n(f) = \{x \in \mathbf{R} : \operatorname{osc} f(x) \ge 2^{-n}\}$ $(n = 1, 2, \ldots)$.

We start with the following lemma:

Lemma 1. *Let $f : \mathbf{R} \to \overline{\mathbf{R}}$ be an upper semicontinuous function (a lower semicontinuous function) such that $f > -\infty$ ($f < \infty$) and $\{x \in \mathbf{R} : f(x) = \infty\}$ ($\{x \in \mathbf{R} : f(x) = -\infty\}$) is nowhere dense. Then for every $c \in \mathbf{R}$ there is an upper semicontinuous (a lower semicontinuous) function $g : \mathbf{R} \to \overline{\mathbf{R}}$ such that $D_n(f) = D_n(g)$ for $n = 1, 2, \ldots, f|C(f) = g|C(g)$, and $c \notin g(\mathbf{R} \setminus C(g))$.*

Proof. Suppose that f is upper semicontinuous. If f is lower semicontinuous, it suffices to consider the function $-f$. Since f and the oscillation of f are upper semicontinuous, all sets $D_n(f)$ $(n = 1, 2, \ldots)$ are closed and nowhere dense. For every $n = 2, 3, \ldots$ there are disjoint finite open intervals I_{nk} with ends belonging to $C(f)$ such that

$$D_n - D_{n-1} = \bigcup_k (D_n \cap I_{nk}).$$

Since every set $D_n \cap I_{nk}$ is compact,

$$2^{-n} \le d_{nk} = \max\{\operatorname{osc} f(t) : t \in D_n \cap I_{nk}\} < 2^{1-n}.$$