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Sets which are Well-Distributed and Invariant Relative to All Isometry Invariant Total Extensions of Lebesgue Measure

1 Introduction

In this paper we discuss subsets A of the real line having the property

$$\mu(A \cap J) = \alpha\ \mu(J), \tag{1}$$

for any interval J of the real line, where $0 < \alpha < 1$ and μ is an isometry-invariant extension of the usual Lebesgue measure λ on the real line. In [18], Simoson considers the notion of a set having this property, but with μ replaced by the Lebesgue outer measure λ^*. Simoson calls such a set a *comb*, and goes on to show that no comb exists. The purpose of this paper is to show that such sets do exist if the outer measure is replaced by suitable extensions of the Lebesgue measure. In particular, for any $\alpha \in (0,1)$, there are sets A, which we shall call *α-shadings* of $\mathbf{R}$, or *combs of shade α*, which have the property that for **any** finitely-additive isometry invariant extension μ of λ to $2^{\mathbf{R}}$, one has

$$\mu(A \cap E) = \alpha\ \lambda(E),$$

for any Lebesgue measurable set E. In fact, many different types of such sets are shown to exist, some having appeared in the literature as examples of non-Lebesgue measurable sets. For instance, one of the classic examples of a non-measurable set is discussed by Halmos [6], and many of the sets in this paper are generalizations of this set. Another set is due to Sierpiṅski [16], which was shown by Hewitt and Stromberg [8] to satisfy $\lambda^*(A \cap J) \geq \frac{1}{2}\lambda(J)$, for intervals $J \subset \mathbf{R}$. Other results concerning some of these sets have been of the form $\lambda^*(A \cap J) = \lambda(J)$, and the reader is referred to Pu [13] and Simoson [19]. The notion of an α-shading will then be generalized to that of