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THE SETS WHERE A FUNCTION HAS INFINITE ONE-SIDED DERIVATIVES

In paper [1] Codyks proved following theorem:

Theorem. Let E_1 and E_2 be disjoint subsets of the set of all real numbers. There exists a function f defined on the set of all real numbers such that $E_1 = \{x : f'(x) = +\infty\}$ and $E_2 = \{x : f'(x) = -\infty\}$ if and only if

(i) E_1 and E_2 are of type $F_{\sigma\delta}$ and of measure zero, and

(ii) There exists disjoint sets F_1 and F_2 of type F_σ such that $E_1 \subset F_1$ and $E_2 \subset F_2$.

In the present paper we consider the problem: Is analogus theorem for left (right) - hand derivative of any finite real function true? It turns out that it is not so. We prove that, for any disjoint sets E_1 E_2 of measure zero there exists a function f such that $E_1 = \{x : f'_-(x) = +\infty\}$ and $E_2 = \{x : f'_-(x) = -\infty\}$. Therefore, exists a function for which the sets $\{x : f'_-(x) = +\infty\}$ and $\{x : f'_-(x) = -\infty\}$ are not-Borel.

We shall apply the following notations:

R - the set of all real numbers;

$R \setminus A$ - the complement of the set A;

A^-, A^+ - the set of all accumulation points of the set A from the left, from the right;

$\overline{f}^-(x), \underline{f}^-(x)$ - the upper left-hand, lower left-hand Dini derivatives of a function f at point x;

$m(A)$ - the Lebesgue measure of a set A;

χ_A - the characteristic function of the set A;

$f'_-(x)$ - the left-hand derivatives of the function f at the point x.