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Separation of points by families of intervals

Let X be a separable metric space. It has been shown (see [1] or [3]) that the Borel structure on X has a minimal generator, i.e. there is a family $\mathscr{F}$ of subsets of X such that the σ–field $\sigma(\mathscr{F})$ generated by $\mathscr{F}$ equals the Borel σ–field $\mathscr{B}(X)$, and such that $\sigma(\mathscr{F}_0) \neq \mathscr{B}(X)$ for any proper sub–collection $\mathscr{F}_0 \subsetneq \mathscr{F}$. Such minimal generators are necessarily countable, as follows from the well–known and easily proved

<u>Lemma 1</u>: Let $(X, \mathscr{B})$ be a measurable space with $\mathscr{B}$ countably generated. If $\mathscr{F} \subsetneq \mathscr{B}$ is such that $\sigma(\mathscr{F}) = \mathscr{B}$, then there is some countable $\mathscr{F}_0 \subsetneq \mathscr{F}$ such that $\sigma(\mathscr{F}_0) = \mathscr{B}$.

In particular, the real line $\mathbb{R}$ has a minimal Borel generator. In [1; p. 19], an argument was made attempting to show that no minimal generator for $\mathbb{R}$ could be constructed using solely intervals. The underlying premise was that a family of intervals is a generator if and only if the set of corresponding interval end–points were dense in $\mathbb{R}$. As pointed out by M. Filipczak [2], this premise is incorrect. Moreover, as we demonstrate, there is indeed a minimal generator for $\mathbb{R}$ comprising only intervals.

Let $\mathscr{F}$ be a family of subsets of a set X. Points $x, y \in X$ are <u>separated</u> by $\mathscr{F}$ if there is some $F \in \mathscr{F}$ such that either

$$x \in F \text{ and } y \notin F \qquad \text{or} \qquad y \in F \text{ and } x \notin F.$$

Say that $\mathscr{F}$ is a <u>minimal</u> <u>separator</u> if $\mathscr{F}$ separates each pair of distinct points from X, but no proper sub–family $\mathscr{F}_0 \subsetneq \mathscr{F}$ does.