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*Geraldo Soares de Souza and G.O. Golightly, Department of Mathematics, Auburn University, Auburn, AL 36849.

A PROOF OF ABEL'S CONTINUITY THEOREM

Let S be the spaces of sequences $s = (s_n)_{n=0}^{\infty}$ of complex terms of convergent series with norm defined by $\|s\|_S = \sup_{n\geq 0} \left|\sum_{j=n}^{\infty} s_j\right|$ and J be the space of all sequences $\beta = (\beta_n)_{n=0}^{\infty}$ of bounded variation with norm $\|\beta\|_J = |\beta_0| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}|$. The following Hölder's type of inequality holds:

If $s = (s_n) \in S$ and $\beta = (\beta_n) \in J$, then $\left|\sum_{n=0}^{\infty} s_n\beta_n\right| \leq \|s\|_S \cdot \|\beta\|_J$.

This may be seen easily by an application of summation by parts.

An interesting application of this inequality is a simple proof of the Stolz form of the Abel Continuity Theorem:

THEOREM: (Abel's Continuity Theorem). If $\sum_{n=0}^{\infty} \alpha_n$ converges and $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, then $\lim_{z\to 1} f(z) = f(1)$, where z is restricted to approach the point 1 in such a way that $|z| < 1$ and $\dfrac{|1-z|}{1-|z|}$ remains bounded.

Proof: First of all let C be a positive absolute constant such that $\dfrac{|1-z|}{1-|z|} \leq C$ and $|z| < 1$. Notice that $\left|\sum_{p=N}^{\infty} \alpha_p z^p\right| \leq \|(z^p)_{p=N}^{\infty}\|_J \cdot \|(\alpha_p)_{p=N}^{\infty}\|_S$ for $N \geq 1$ and by the above inequality applied to the sequences (α_p) and (z^p), since $(z^p)_{p=N}^{\infty} \in J$; in fact, since $|z| < 1$