The Powers and their Bernstein Polynomials

We wish to establish a relationship between the powers and their Bernstein polynomials.

We start with a brief review of Stirling numbers because of their importance in developing the theory.

Denote Stirling numbers of the first kind by $s(m,r)$.

$$\begin{cases} s(1,1) = 1 \text{ while } s(1,r) = 0 \text{ for } r \neq 1, \\ s(m,r-1) - ms(m,r) = s(m+1,r). \end{cases} \quad (1)$$

Denote Stirling numbers of the second kind by $S(m,r)$.

$$\begin{cases} S(1,1) = 1 \text{ while } S(1,r) = 0 \text{ for } r \neq 1, \\ S(m,r-1) + rS(m,r) = S(m+1,r). \end{cases} \quad (2)$$

We write them in matrix form: $A = (s(m,r))$, $S = (S(m,r))$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & -3 & 1 \\ -6 & 11 & -6 & 1 \\ 24 & -50 & 35 & -10 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 7 & 6 & 1 \\ 1 & 15 & 25 & 10 & 1 \end{pmatrix} \quad (3)$$

They are inverse to each other.

$$A^{-1} = S, \quad S^{-1} = A, \quad AS = SA = I \quad (4)$$

Some mathematicians define Stirling numbers of the second kind to be

$$S(m,r) = \frac{1}{r!} \sum_{k=1}^{r} (-1)^k \binom{r}{k} (r-k)^m = \frac{(-1)^r}{r!} \sum_{k=1}^{r} (-1)^k \binom{r}{k} k^m \quad (5)$$