SYMMETRIC REAL ANALYSIS: A SURVEY

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Section 1: Introduction

This survey concerns what we call, for lack of a better name, "symmetric real analysis." Under the heading of symmetric real analysis, we include such topics as symmetric differentiation, symmetric continuity, symmetric functions, smooth functions and locally symmetric sets. In short, this name is a catch-all phrase for any definition or property which is intrinsically based upon a symmetric difference of some order.

All of the examples of symmetric real analysis mentioned above originate in the study of trigonometric series. For this reason, they play an important role in the classical works of Riemann [Ri], Lebesgue [Le], Fatou [Fa] and others. Since their inception, however, they have been used in such areas as approximation theory and harmonic analysis. This survey will not be concerned with these applications of symmetry. Rather, we will examine the behavior of functions satisfying certain symmetry properties which have proved useful, mostly in the study of the pointwise convergence of trigonometric series. Any reader interested in the application of these ideas to Fourier series may refer to Hobson [Ho, Vol. 2, Ch. 8] and Zygmund [Zy, Vol. 2, Ch. 11].

For a function $f:R \to R$, we define the n'th symmetric difference of f at x to be

$$\Delta^n f(x,h) = \Sigma_{i=0}^{n}(-1)^i \binom{n}{i} f(x + (n - 2i)h). \tag{1}$$

Of particular importance are the first and second symmetric differences of f,

$$\Delta f(x,h) = \Delta^1 f(x,h) = f(x + h) - f(x - h)$$

and

$$\Delta^2 f(x,h) = f(x + h) + f(x - h) - 2f(x).$$

A function f is symmetrically continuous at x iff

$$\lim_{h \to 0} \Delta f(x,h) = 0.$$

It is a-smooth ($a \geq 0$) iff $\lim_{h \to 0} \Delta^2 f(x,h)/h^a = 0$. In the special cases when a = 0 or