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SOME THEOREMS ON DINI DERIVATES

The relationships between the density of sets of points where the various Dini derivates of a function are nonnegative are studied.

Theorem 1: If $f(x)$ is a real valued function of a real variable, λ any real number and $\{ x : D_-f(x) \geq \lambda \}$ is dense, then $\{ x : D^+f(x) \geq \lambda \}$ is dense.

Proof: Without loss of generality, assume $\lambda=0$. Let (a,b) be any interval. There exists x_1 in (a,b) such that $D_-f(x_1) > -1$. Therefore, there is some $\delta_1 > 0$ such that for every t in $(x_1-\delta_1,x_1)$,

$f(t) < f(x_1) + (x_1-t)$.

Choose $\delta_1 < 1$ and such that $x_1-\delta_1 > a$. There is x_2 in $(x_1-\delta_1,x_1)$ such that $D_-f(x_2) > -1/2$. Therefore, there is some $\delta_2 > 0$ such that for every t in $(x_2-\delta_2,x_2)$,

$f(t) < f(x_2) + (1/2)(x_2-t)$.

Choose $\delta_2 < 1/2$ and such that $x_2-\delta_2 > x_1-\delta_1$.

Continuing in this manner we obtain a decreasing sequence of intervals $\{(x_n-\delta_n,x_n)\}$ such that $\delta_n < 1/n$.