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CONVERSION FORMULAS FOR THE LEBESGUE-STIELTJES INTEGRAL

0. Introduction

We present here some formulas converting Lebesgue-Stieltjes integrals with a continuous integrator h of bounded variation into Lebesgue integrals over the range of h. A special case is Banach's indicatrix formula displayed as (10) below. Indeed, we use an extension of Banach's formula to prove the general conversion formulas. One of these formulas (6) in conjunction with (8) and the Fubini theorem for the generalized Riemann integral [11] provides a handy proof of Green's theorem. In general all our integrals are defined by Kurzweil-Henstock integration using endpoint tags in the approximating sums. [3], [4], [5] ,[6], [7], [10] and [11] But wherever we have absolute integrability our integrals here are equivalent to Lebesgue-Stieltjes integrals. Our use of differentials is based on the concepts introduced in [6], [7] and [8].

We begin with some relevant definitions. A *cell* is a closed interval $K = [a, b]$ in $\mathbb{R}$ with $a < b$. A *figure* is a finite union of disjoint cells. The *indicator* 1_E of a subset E of $\mathbb{R}$ is the function on $\mathbb{R}$ with value 1 on E and 0 on the complement $\mathbb{R} \setminus E$. E^0 is the interior of E. For h a function on $K = [a, b]$ we define $\Delta h(K) = h(b) - h(a)$. For h continuous and of bounded variation on K, a subset E of K is *dh-measurable* if the differential $1_E\, dh$ is integrable over K. This is equivalent to the existence of the Lebesgue-Stieltjes integral $\int_E dh$. A dh-measurable set differs from a Borel set by a dh-null set. (See [7].) The *variation function* for h is the function v defined on K by

$$(0) \qquad v(t) = \int_a^t |dh(s)|.$$

v is characterized by the conditions $v(a) = 0$ and $dv = |dh|$.

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