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NOTE ON POINT SET THEORY

A large number of analogies between Baire category and Lebesgue measure are unified and generalized in [3]. Here an additional analogy established in [5] is generalized to perfect category bases $(X,\mathcal{C})$, where X is a dense-in-itself complete metric space. For definitions and properties used below refer to [1]-[4].

Theorem. For any given sequence of Baire sets, there exists in each abundant Baire set a denumerable set which cannot be represented as the limit of any subsequence of the given sequence.

Proof. Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a given sequence of Baire sets and let S be an abundant Baire set. According to the Fundamental Theorem, there exists a region A in which S is abundant everywhere. By Theorem 1.III.2 of [3] we have

$$A - S = \bigcup_{i=1}^{\infty} T_i$$

where each set T_i is a singular set. We proceed to determine a dyadic schema of subregions A_σ of A, where σ varies over all finite sequences of elements of the set $\mathbb{B} = \{0,1\}$.

Define A_0 and A_1 to be two disjoint subregions of A each of which has diameter ≤ 1 and is disjoint from the set T_1. For fixed $\beta \in \mathbb{B}$ we denote by $R_{\beta,1}$ the first one of the sets E_1, $X - E_1$ which is abundant in A_β and choose a subregion C_β of A_β in which $R_{\beta,1}$ is abundant everywhere. Since $R_{\beta,1}$ is a Baire set we have

$$C_\beta - R_{\beta,1} = \bigcup_{i=1}^{\infty} T_{\beta,i}$$

where each set $T_{\beta,i}$ is singular. We then define $A_{\beta 0}$ and $A_{\beta 1}$ to be two disjoint subregions of C_β each of which has diameter $\leq \frac{1}{2}$ and is disjoint from T_1, T_2, $T_{\beta,1}$ and $T_{\beta,2}$.