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Jozef Doboš, Katedra matematiky VŠT, Švermova 9, 040 01 Košice, Czechoslovakia

ON EXTREMAL VALUES OF CONTINUOUS MONOTONE FUNCTIONS

The notion of monotone for continuous functions was introduced by G.T. Whyburn [3]. The purpose of this note is to investigate extremal values of continuous monotone functions of several variables.

Definition. (Also see [2].) Let X be a topological space. A function $f : X \to R$ is said to be monotone if for each point $y \in R$, $f^{-1}(y)$ is connected in X.

Lemma. (See [1].) Let X be a locally connected space. Then $f : X \to R$ is continuous if and only if f has the Darboux property and there is a dense set $P \subset R$ such that $f^{-1}(p)$ is closed for each $p \in P$. (Recall that f is said to have the Darboux property if it maps connected sets to connected sets.)

Proposition. Let X be a locally connected space. Let $f : X \to R$ be a monotone function with the Darboux property. Then f is continuous.

Proof. Let $y \in R$. Suppose that $x \in Cl\, f^{-1}(y)$. Then $A = f^{-1}(y) \cup \{x\}$ is a connected set in X. Thus $f(A)$ is connected in R. Hence $f(x) = y$. Therefore $f^{-1}(y)$ is closed.

Theorem. Let X be a T_3-space without isolated points. Suppose that for each $x \in X$ there is a base $\mathcal{B}(x)$ of open neighborhoods of x such that for each $B \in \mathcal{B}(x)$ the sets B, $X - B$ are connected and $Fr\, B$ is compact (where $Fr\, T = Cl\, T - Int\, T$). Let $f : X \to R$ be a monotone function with the Darboux property. Then f has a strict absolute extremum at any point a where f has a strict relative extremum.

Proof. Suppose f has a strict relative maximum at $a \in X$. (The second case is similar.) Let $b \in X$ such that $f(a) \leqq f(b)$. Then there is an open neighborhood U of a such that

(1) $$\forall \ x \in U, \ x \neq a : f(x) < f(a).$$

Since a is an accumulation point of X, there is $c \in U - \{a\}$. Since X is T_3, there is a closed neighborhood V of a such that $V \subset U - \{c\}$. By the assumption there is an open neighborhood A of a such that $A \subset V$, the sets A, $X - A$ are