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MARTINGALE PROOF OF THE EXISTENCE OF LEBESGUE POINTS

The usual proof of the existence of Lebesgue points of a summable function is via Vitali's covering theorem or its modifications. We give here an alternative proof which reduces geometric considerations to a very simple lemma. Our proof is based on Lévy's martingale convergence theorem.

Let $\mathcal{A}$ be a family of sets and let X be any set. Then $\mathcal{A}|X = \{A \cap X : A \in \mathcal{A}\}$. By $\sigma\mathcal{A}$ we denote the σ-field generated by $\mathcal{A}$. Let $\mathbb{Z}$ be the set of all integers. The symbol χ_A will stand for the characteristic function of a set A. The n-dimensional Lebesgue measure in $\mathbb{R}^n$ will be denoted by λ (the same symbol λ will be used for each positive integer n). For any measure space $(\Omega, \mathcal{F}, \mu)$ we shall denote by $L_1(\Omega)$ the family of real functions f measurable with respect to $\mathcal{F}$ such that $\int_\Omega |f| \, d\mu < \infty$. If Ω is an open subset of $\mathbb{R}^n$, $\mathcal{F}$ will be the family of Lebesgue measurable sets and $\mu = \lambda$. For a subset X of $\mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$ put $x + X = \{x + y : y \in X\}$.

Let $f \in L_1(U)$, where U is an open subset of $\mathbb{R}^n$. We say that $x \in U$ is a *Lebesgue point* of f if