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A DESCRIPTIVE CHARACTERIZATION OF THE GENERALIZED RIEMANN INTEGRAL

A function f is Denjoy-Perron integrable on $[a, b]$ if and only if there exists an ACG_* function F on $[a, b]$ such that $F' = f$ almost everywhere on $[a, b]$. In this paper we present a similar result (Theorem 4) for the generalized Riemann integral using a different notion of absolute continuity. See also the paper by C. Seng in this volume.

We will assume familiarity with the definitions of the Denjoy-Perron and generalized Riemann integrals. Throughout this paper $\mathcal{P}$ will denote a finite collection of non-overlapping tagged intervals in $[a, b]$. For $\mathcal{P} = \{(t_i, [c_i, d_i]) : 1 \le i \le N\}$, we will write

$$f(\mathcal{P}) = \sum_{i=1}^{N} f(t_i)(d_i - c_i), \quad F(\mathcal{P}) = \sum_{i=1}^{N} \big(F(d_i) - F(c_i)\big), \quad \text{and} \quad \mu(\mathcal{P}) = \sum_{i=1}^{N} (d_i - c_i).$$

This is an abuse of notation, but it is quite convenient. Let δ be a positive function defined on $[a, b]$. We say that $\mathcal{P}$ is subordinate to δ if $[c_i, d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for each i and that $\mathcal{P}$ is subordinate to δ on $[a, b]$ if in addition $\mathcal{P}$ is a partition of $[a, b]$. Given a set E and a point t, let $\rho(t, E)$ be the distance from t to E, $\mathcal{C}E$ be the complement of E, and $\overline{E}$ be the closure of E.

DEFINITION 1: Let $F : [a, b] \to R$ and let $E \subset [a, b]$. The function F is AC_δ on E if for each $\epsilon > 0$ there exist a positive number η and a positive function δ on E such that $|F(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is subordinate to δ, all of the tags of $\mathcal{P}$ are in E, and $\mu(\mathcal{P}) < \eta$. The function F is ACG_δ on E if E can be written as a countable union of sets on each of which the function F is AC_δ.

LEMMA 2: Suppose that $F : [a, b] \to R$ is ACG_δ on $[a, b]$ and let $E \subset [a, b]$. If $\mu(E) = 0$, then for each $\epsilon > 0$ there exists a positive function δ on E such that $|F(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is subordinate to δ and all of the tags of $\mathcal{P}$ are in E.

PROOF: Let $E = \cup_n E_n$ where the E_n's are disjoint and F is AC_δ on each E_n. Let $\epsilon > 0$. For each n there exist a positive function δ_n on E_n and a positive number η_n such that $|F(\mathcal{P})| < \epsilon/2^n$ whenever $\mathcal{P}$ is subordinate to δ_n, all of the tags of $\mathcal{P}$ are in E_n, and $\mu(\mathcal{P}) < \eta_n$. For each n choose an open set O_n such that $E_n \subset O_n$ and $\mu(O_n) < \eta_n$. Let $\delta(t) = \min\{\delta_n(t), \rho(t, \mathcal{C}O_n)\}$ for $t \in E_n$. Suppose that $\mathcal{P}$ is subordinate to δ and that all of the tags of $\mathcal{P}$ are in E. Let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that has tags in E_n. Note that $\mu(\mathcal{P}_n) < \eta_n$ and compute

$$|F(\mathcal{P})| \le \sum_n |F(\mathcal{P}_n)| < \sum_n \epsilon/2^n < \epsilon.$$

This completes the proof.