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THE RADON-NIKODYM DERIVATIVE IN EUCLIDEAN SPACES

The usual Radon-Nikodym Theorem can be stated as follows: If Φ is a signed measure and m is a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X where both Φ and m are σ-finite, then there is an m-measurable function f and a singular completely additive function of a set Θ such that for each $E \in \mathcal{A}$, $\Phi(E) = \int_E f\, dm + \Theta(E)$. Here Θ is singular means that there is a set $Z \in \mathcal{A}$ with $m(Z) = 0$ such that for $B \in \mathcal{A}$ with $B \cap Z = \phi$, $\Theta(B) = 0$. The function f is sometimes called the Radon-Nikodym derivative of Φ with respect to m and written $d\Phi/dm$.

In Euclidean spaces, if a function Φ is defined on the Borel sets, the general upper derivate is defined by $\bar{D}\Phi(x) = \sup \lim \Phi(E_n)/m(E_n)$ where the supremum is taken over all regular sequences $\{E_n\}$ of closed sets with $x \in \bigcap E_n$ and $\lim \operatorname{diam} E_n = 0$ for which $\lim \Phi(E_n)/m(E_n)$ exists. A sequence $\{E_n\}$ is regular provided there is $r > o$ such that for each n $m(E_n)/m(Q_n) > r$ where Q_n is the smallest cube containing E_n. The general lower derivate $\underline{D}\Phi$ is defined by the infimum of such limits and the general derivative $D\Phi(x)$ is the common value of $\bar{D}\Phi(x)$ and $\underline{D}\Phi(x)$ when they are equal.