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ANOTHER PROOF OF THE MEASURABILITY OF δ FOR THE GENERALIZED RIEMANN INTEGRAL

The purpose of this paper is to show that restricting the function δ in the generalized Riemann integral to be measurable does not change the nature of the integral. The two definitions that follow will clarify the problem.

DEFINITION 1: Let $\delta(\cdot)$ be a positive function defined on the interval $[a,b]$. A tagged interval $(s,[c,d])$ consists of an interval $[c,d]$ in $[a,b]$ and a point s in $[c,d]$. The tagged interval $(s,[c,d])$ is subordinate to δ if $[c,d] \subset (s-\delta(s), s+\delta(s))$. Let $\mathcal{P} = \{(s_i,[c_i,d_i]) : 1 \leq i \leq N\}$ be a finite collection of non-overlapping tagged intervals in $[a,b]$. If $(s_i,[c_i,d_i])$ is subordinate to δ for each i, then we write $\mathcal{P}$ is subordinate to δ. If in addition $\mathcal{P}$ is a partition of $[a,b]$, then we write $\mathcal{P}$ is subordinate to δ on $[a,b]$. For a function $f:[a,b] \to R$ and a function F defined on the intervals of $[a,b]$, we write

$$f(\mathcal{P}) = \textstyle\sum_i f(s_i)(d_i - c_i) \quad \text{and} \quad F(\mathcal{P}) = \sum_i F([c_i,d_i]).$$

DEFINITION 2: The function $f:[a,b] \to R$ is GR (mGR) integrable on $[a,b]$ if there exists a real number α with the following property: for each $\epsilon > 0$ there exists a positive (positive, measurable) function δ on $[a,b]$ such that $|f(\mathcal{P}) - \alpha| < \epsilon$ whenever $\mathcal{P}$ is subordinate to δ on $[a,b]$. The function f is GR (mGR) integrable on the set $E \subset [a,b]$ if $f\chi_E$ is GR (mGR) integrable on $[a,b]$.

It is clear that every mGR integrable function is GR integrable and that the integrals are equal. We will show that every GR integrable function is mGR integrable. We first establish some notation. Given a point t and a set E, CE is the complement of E, $\mu(E)$ is the Lebesgue measure of E, χ_E is the characteristic function of E, and $\rho(t,E)$ is the distance from t to E. We will use $\omega(f,I)$ to denote the oscillation of the function f on the interval I.

The mGR integral shares many of the properties of the GR integral, including integrability on subintervals and Henstock's Lemma. By easy adaptations of the proofs for the GR integral, we obtain the next two results.

THEOREM 3: If $f:[a,b] \to R$ is mGR integrable on each of the intervals $[a,c]$ and $[c,b]$, then f is mGR integrable on $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

THEOREM 4: Suppose that $f:[a,b] \to R$ is mGR integrable on each interval $[\alpha,\beta] \subset (a,b)$. If $\int_\alpha^\beta f$ converges to a finite limit as $\alpha \to a^+$ and $\beta \to b^-$, then f is mGR integrable on $[a,b]$ and $\int_a^b f = \lim_{\substack{\alpha \to a^+ \\ \beta \to b^-}} \int_\alpha^\beta f$.