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ON DISCONTINUITY POINTS FOR CLOSED GRAPH FUNCTIONS

We say that a function f from a space X into a space Y has a closed graph if the graph of the function f, i.e. the set $\{(x,y) \in X \times Y; y = f(x)\}$ is a closed subset of the product $X \times Y$. We denote by C_f (D_f) the set of all points at which the function f is continuous (discontinuous).

There are many papers which deal with the set D_f for closed graph functions. (See for example [1], [2] or [4].) The purpose of the present paper is to continue the investigation of this set.

Proposition A. (See [4].) Let $I \subset R$ be an interval. Then for each closed graph function $f : I \to R$ the set D_f is closed and nowhere dense.

Proposition B. (See [1].) Let $f : X \to R^n$ have a closed graph, where X is a Hausdorff space. Let $x \in D_f$. Then f is unbounded in every neighborhood of the point x.

Theorem 1. Let $f : I \to R$ have a closed graph, where $I \subset R$ is an interval. Let $x \in D_f$. Then for each neighborhood U of x there is an interval $J \subset U \cap C_f$ such that f is unbounded on J.

Proof. Suppose to the contrary that there is a $\delta > 0$ such that for each interval $J \subset (x - \delta, x + \delta) \cap I \cap C_f$ the function f is bounded on J. Put $F = [x - \delta/2, x + \delta/2] \cap I \cap D_f$. Since f is a Baire class one function (See [4].), there is an $x_0 \in F$ such that the function $f|_F$ is continuous at x_0. Put $V = (x - \delta, x + \delta) \cap I \cap C_f$. Since V is open in I, there is a countable family J of pairwise disjoint open intervals such that $V = \bigcup J$. Since $x_0 \in D_f$, the function f is unbounded in each neighborhood of x_0. Thus there is a monotone sequence $\{x_n\}$ of points $x_n \in U$ such that $x_n \to x_0$ and the sequence $\{f(x_n)\}$ is unbounded. Suppose that $x_n < x_0$ for each $n = 1, 2, \dots$. (The opposite case is similar.) Then for each n there is a $J_n \in J$ such that $x_n \in J_n$. Let $J_n = (a_n, b_n)$. Then $x_n < b_n \le x_0$ for each $n = 1, 2, \dots$. Since f has a closed graph and it is by assumption bounded on each J_n, the function $f|_{\overline{J_n}}$ is continuous. Since $f|_F$ is continuous at x_0, it follows that $f(b_n) \to f(x_0)$. From the Darboux property