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ON THE DANIÉLL INTEGRAL

H. M. MacNeille in [2] suggested the following simple definition of Lebesgue integrable functions:

A real function f (defined on $\mathbf{R}^N$) is Lebesgue integrable if there exists a sequence of step functions $f_1, f_2, f_3, \ldots$ such that

(*) $\sum_{n=1}^{\infty} \int |f_n| < \infty,$

(**) $f(x) = \sum_{n=1}^{\infty} f_n(x)$ *for every* $x \in \mathbf{R}^N$ *for which* $\sum_{n=1}^{\infty} |f_n(x)| < \infty.$

Systematic use of the above approach in [3] shows that it gives a very fast and natural way of developing the theory of Lebesgue integral as well as the Bochner integral. In this note we show that the same method can be successfully used to construct the Daniell integral. In the standard approach one has to introduce auxiliary spaces of the so-called "over-functions" and "under-functions" which are used only as a step in the construction and are not needed later. In the present method the integral and the space of integrable functions are introduced in one step without any additional constructions. Moreover, our approach simplifies proofs of important theorems on the integral.

Some proofs in this note are adaptations of proofs of corresponding properties of the Lebesgue or Bochner integral in [3]. Other proofs are actually simpler in the abstract setting of the Daniell integral. Since the aim of this note is to show that the described approach simplifies the construction of the Daniell integral no proofs are omitted.

It is interesting that although the Daniell integral is discussed in [3] the traditional approach is used.

A triple $(K, \mathcal{U}, \int)$ will be called a *Daniell space* if K is a nonempty set, $\mathcal{U}$ is a Riesz space (i.e., vector lattice) of real valued functions on K, and $\int$ is a real linear functional on $\mathcal{U}$ such that

(1) $\int f \geq 0$ whenever $f \geq 0$,

(2) $\int f_n \to 0$ for every non-increasing sequence of functions $f_n \in \mathcal{U}$ such that $f_n(x) \to 0$ for every $x \in K$.

For the lattice operations we use the following notation: $f \cup g = max(f, g)$ and $f \cap g = min(f, g)$.