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VARIATIONS ON PRODUCTS AND QUOTIENTS OF DARBOUX FUNCTIONS

I. Let us establish some of the terminology to be used. R denotes the real line and N denotes the set of natural numbers. If $a,b\in R$, then (a,b) denotes the open interval with the end-points a,b. For $A\subset R$, we shall say that I is an open interval of A iff $I=(a,b)\cap A$ for some $a,b\in R$. If B is a planar set, we shall denote its x - projection by dom B and its y - projection by rng B. If A,B are subsets of R then $A\ B=\{a\cdot b\ :\ a\in A,\ b\in B\}$, $a\ B=\{a\}\cdot B$ and $A^{-1}=\{1/a\ :\ a\in A\setminus\{0\}\}$. For $A\subset R$, $a\in R$, and $f:A\to R$, we define the set $[f<a]$ as $\{x\in A : f(x)<a\}$. Analogously, we define the sets $[f>a]$ and $[f=a]$. Let $A\subset R$ be a c-dense set in itself (where c denotes the cardinality of the continuum) and let B be a subset of R. We say that $f:A\to B$ is an (A,B)-Darboux function iff f has the intermediate value property, i.e. $(f(x),f(y))\cap B\subset f((x,y)\cap A)$ for each $x,y\in A$. Let $\mathcal{D}(A,B)$ denote the class of all (A,B)-Darboux functions. Let $\mathcal{D}^*(A,B)$ denote the class of all functions $f:A\to B$ which take on every $y\in B$ in every non-empty interval I of A. Let $\mathcal{D}^{**}(A,B)$ denote the class of all functions $f:A\to B$ which take on every $y\in B$ c times in every interval of A. It is clear that $\mathcal{D}^{**}(A,B)\subset\mathcal{D}^*(A,B)\subset\mathcal{D}(A,B)$ for every bilaterally c-dense subset A of R and every subset B of R. For A=B=R, we shall denote the classes $\mathcal{D}(A,B)$, $\mathcal{D}^*(A,B)$, and $\mathcal{D}^{**}(A,B)$ by $\mathcal{D}$, $\mathcal{D}^*$, $\mathcal{D}^{**}$ (see [3]).

A.M. Bruckner and J. Ceder proved the following theorem.

THEOREM 1. [3]. Let $f\in\mathcal{D}$ be constant on no subinterval of R and let M be a set of real numbers whose complement is dense. Then for each countable dense subset D of $R\setminus M$ there exists a function $d\in\mathcal{D}^*$ such that the range of $f+d$ is D.