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THE ω-LIMIT SETS FOR SELF MAPS OF AN INTERVAL

Let f be a function mapping $I = [0,1]$ into itself. A set $K \subset I$ is called an ω-limit set for f if there exists $x \in I$ such that K is the cluster set of the sequence $\{f^n(x)\}$. (Here, as usual, $f^1 = f$ and $f^{n+1} = f \circ f^n$ for $n = 1,2,3,$ --.) We write $\omega_f(x) = K$ to indicate K is the ω-limit set of x under f.

For sufficiently well-behaved functions one finds that either there is a single set K that serves as the ω-limit set for almost all $x \in I$, or there is some form of chaotic behavior.

For the typical continuous f, no single set can serve as an ω-limit set for almost all $x \in I$. In fact [ABL], there exists a set K of full measure such that for every $x \in K$, $\omega_f(x)$ is a Cantor set K_x -- but the sets K_x are distinct and pairwise disjoint: if $x \neq y$, then $K_x \cap K_y = \phi$.

An ω-limit set must be closed, but beyond that, what restrictions must apply? On the one hand, one finds in the literature all sorts of closed sets that can serve as ω-limit sets for a continuous function. Finite sets give rise to periodic behavior, Cantor sets arise in various ways, as do countable closed sets and even intervals. On the other hand, an ω-limit set with interior must consist of a finite union of intervals.

In addition, Šarkovskii [S], indicates that if an infinite ω-limit set contains an isolated point, it must contain infinitely many isolated points.

This assertion of Šarkovskii's led several of us [ABCP] to try to obtain a characterization of those sets that can serve as ω-limit sets of continuous functions.

Since no proof of Šarkovskii's assertion appeared in [S], we tried first to prove this assertion. But we were unable to rule out a certain scenario that led to a revealing example.