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LIMITS UNDER THE INTEGRAL SIGN[1]

Using a decomposable division space, we study

$$(1) \qquad \lim_{n\to\infty} \int_E f_n \, dm = \int_E \lim_{n\to\infty} f_n \, dm ,$$

where the f_n are functions of points with values in a space K, and m is a function of interval-point pairs with values in K or real or complex scalars, so that the values of f_n and m can be multiplied together. When K is linear with real scalars a and a norm $||k||$ satisfying $||ak|| = |a|\cdot||k||$, it is usual to have properties (i) $V(m;A;E) < \infty$, (ii) $||f_n - f|| \to 0$ m-almost everywhere in E, (iii) Fm and $F_n m$ $(n = 1,2, \ldots)$ integrable on E, and (iv) $m \geq 0$ and $||f_n|| \leq F$ $(n = 1,2, \ldots)$. These are a type of Arzela-Lebesgue condition in K. But (i) and (iv) restrict the test; (i) cuts out many applications to Feynman integration. Again, not all topological groups have even a group norm, while the restriction to $f_n(t)m(I,t)$ is a weakness. Generalizing to $h_n(I,t)$, we have a problem highlighted by the following examples.

On the real line, for each fixed integer $j \geq 2$ let $h_j([u,v),t) = v-u$ if $(j+1)u/j < v \leq ju/(j-1)$ $(u > 0)$, and otherwise let $h_j = 0$. Then $h = \sum_{j=2}^{\infty} h_j = v-u$ $(u < v \leq 2u)$ and otherwise $h = 0$, and the guage integrals $\int_{[0,1)} dh_j = 0$ (all $j \geq 2$), $\int_{[0,1)} dh = 1$, and $\sum_{j=2}^{\infty} h_{2j}$ is not integrable over any interval of $[0,1)$.

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