Fractional Hadamard Powers of Positive Definite Matrices

by

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1. Preamble.

Let $P(t) = (a_{ij}{}^t)$ be any n x n real symmetric matrix, where $a_{ij} \geq 0$ and $t \geq 1$. We consider the following conjecture.

Conjecture. If P(1) is positive definite then P(t) is also positive definite for all $t \geq 1$. (C)

This conjecture is relevant to the study of positive definite similarity matrices, such as correlation matrices, which arise in the analysis of psychological data. For such matrices, monotonic transformations of the coefficients are sought whose properties include the preservation of both the ordering of the coefficients and the positive definiteness of the matrices.

The conjecture (C) is always true when n = 2 or 3 and, in certain special cases, for all n. It is in general untrue when n = 4, and it is thus in general untrue when $n \geq 4$ since a real symmetric matrix is positive definite if and only if all of its principal minors are strictly positive.

Remark. Since $P(t) = (a_{ij}{}^t)$ is positive definite if and only if the real quadratic form $x^TP(t)x$ is positive definite, where $x = (x_i)$, the substitution $x_i = y_i/\sqrt{a_{ii}}$ shows that it will suffice to assume that $a_{ii} = 1$ for all i, and that $0 \leq a_{ij} < 1$ when $i \neq j$. (1.1)

2. Cases when the conjecture (C) is true.*

Let $P(t) = (a_{ij}{}^t)$ be any n x n real symmetric matrix which satisfies (1.1), let $t \geq 1$ and let P(1) be positive definite. Then P(t) is also positive definite in the following cases:

1. for all $t \geq 1$ when n = 2 and when n = 3;
2. for all n when t is any positive integer;
3. for all n provided that $t \geq T$, where the value of T depends upon the particular matrix;
4. for all $t \geq 1$ and for all n when $a_{ij} = \alpha_i\alpha_j$ and $\alpha_i > 0$, for $i \neq j$ and i,j = 1,2,....,n;
5. for all t and any given n if and only if P(t) is positive definite whenever $1 < t < 2$. (2.1)

Proofs. All of the proofs are elementary. In some of them use is made of the facts that a real symmetric matrix is positive definite if and only if all of its principal minors are strictly positive and that the Hadamard product of two positive definite matrices is also positive definite.

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* Some work in this section is due to AM Russell.