Density Continuous Functions

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Let $\mathbf{R}_O$ and $\mathbf{R}_D$ denote the real numbers with the ordinary topology and the density topology, respectively. There are four ways that a function can be continuous using these two topologies. They are denoted by

$$C_{XY} = \{f:\mathbf{R}_X \to \mathbf{R}_Y\}.$$

So, for example, C_{OO} is the set of functions continuous in the usual sense and C_{DO} is the set of approximately continuous functions. It is not hard to prove that C_{OD} consists precisely of the constant functions. The functions in C_{DD} are called the *density continuous* functions. It is also clear from the definitions that

(1) $$C_{OD} \subset C_{OO} \subset C_{DO} \supset C_{DD}.$$

Of course, the ordinary continuous and the approximately continuous functions have been studied extensively, but less is known about the density continuous functions. It turns out that their structure is more difficult than the other two classes, as the following statements show.

Example 1. ([CLO1]) There exists a C^∞ function which is not density continuous.

Example 2. ([CL1]) There is an $f \in C_{DD} \cap C^\infty$ such that $f(x) + x \notin C_{DD}$. Therefore, C_{DD} is not closed under pointwise addition.

Example 3. ([KO1]) C_{DD} is not closed under uniform convergence, or even equal convergence.

In fact the techniques used in these examples can be used to establish the following theorem.

Theorem 1. ([CLO1; CLO2]) With the uniform topology, C_{DD} is first category in itself and $C_{DD} \cap C_{OO}$ is first category in C_{OO}.

Since, as noted above, C_{OD} only contains the constant functions, Theorem 1 implies that first and third containments in equation (1) are proper and first category. (That the second containment is proper and first category is well known.) This can be construed to mean that C_{DD} is a small space in the topological sense, mak-