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ON THE GENERAL THEORY OF POINT SETS, II.

In the present article we state the most important open problems of the general theory and we indicate to what extent analogies between Baire category and Lebesgue measure are valid for other classifications of sets. Specifically, we shall enunciate general theorems true of all six primal examples discussed in [19], concerning Baire category, Lebesgue measure, Hausdorff measure, Hausdorff dimension, topological dimension, and Marczewski's classification.

We recall that the foundation of the general theory is the axiomatically defined notion of a category base. A pair $(X,\mathcal{C})$, where X is a nonempty set and $\mathcal{C}$ is a family of subsets of X, is called a category base if the nonempty sets in $\mathcal{C}$, called regions, satisfy the following axioms:

1. Every point of X belongs to some region; i. e. $X = \bigcup\mathcal{C}$
2. Suppose A is a region and $\mathcal{D}$ is a nonempty family of disjoint regions which has power less than the power of $\mathcal{C}$.
   a. If $A\cap(\bigcup\mathcal{D})$ contains a region then there is a region $D\in\mathcal{D}$ such that $A\cap D$ contains a region.
   b. If $A\cap(\bigcup\mathcal{D})$ contains no region then there is a region $B\subset A$ which is disjoint from every region in $\mathcal{D}$.

With respect to a given category base $(X,\mathcal{C})$ the subsets of X are classified as follows: A set S is singular iff every region contains a subregion disjoint from S. A countable union of singular sets is called a meager set. A set which is not meager is called an abundant set. A set is called a Baire set iff every region contains a subregion whose intersection with either the set or its complement is meager.

We recall

(I) The singular sets form an ideal and the meager sets form a σ-ideal.

(II) The Baire sets form a σ-field containing all regions and all meager sets.

(III) The family of Baire sets is closed under operation $\mathcal{A}$.